GRADED CELLULAR BASES FOR TEMPERLEY-LIEB ALGEBRAS OF TYPE A AND B.

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ABSTRACT. We show that the Temperley-Lieb algebra of type A and the blob algebra (also known as the Temperley-Lieb algebra of type B) at roots of unity are $\mathbb{Z}$-graded algebras. We moreover show that they are graded cellular algebras, thus making their cell modules, or standard modules, graded modules for the algebras.

1. INTRODUCTION.

In this paper we study the Temperley-Lieb algebra $Tl_n(q)$. It was introduced around forty years ago from considerations in statistical mechanics, but has since turned out to be related to many topics of mathematics as well, including knot theory, operator theory, algebraic combinatorics and algebraic Lie theory. As of today, it is an object well known to a general audience in physics as well as mathematics and at the same time it remains at the center of a big number of research articles being published each year in both areas.

Our main emphasis lies on a two-parameter generalization $b_n(q,m)$ of the Temperley-Lieb algebra that was introduced by P. Martin and H. Saleur in [13], as a way of introducing periodicity in the physical model. An important feature of both $Tl_n(q)$ and $b_n(q,m)$ is the fact that they are diagram algebras, that is they have bases parameterized by certain planar diagrams, such that the multiplications are given by concatenation of these diagrams. In the case of $Tl_n(q)$ these diagrams are the socalled *bridges* or *Temperley-Lieb diagrams*, in the case of $b_n(q,m)$ the diagrams are certain marked Temperley-Lieb diagrams and for this reason $b_n(q,m)$ was called the blob algebra in [13].

We are interested in the non-semisimple representation theory of $Tl_n(q)$ and $b_n(q,m)$, which is the case where q is specialized at a root of unity. The $Tl_n(q)$-case is connected via Schur-Weyl duality to the representation theory of the quantum group associated with SL_2. The $b_n(q,m)$-case is more intriguing and has received quite a lot of attention over the last decade. It has been shown to share a surprisingly number of properties with objects that normally arise in Lie theory. In particular, it was shown in [15] that the decomposition numbers are given by evaluations at 1 of certain Kazhdan-Lusztig polynomials associated with an infinite dihedral Weyl group.

The fact that the decomposition numbers for $b_n(q,m)$ come from polynomials gives a first indication of the existence of a $\mathbb{Z}$-graded structure on $b_n(q,m)$ and on its standard modules, and indeed a main goal of our paper is to construct such a graded structure on $b_n(q,m)$.

A main input to our paper comes from the seminal work of Brundan and Kleshchev that constructs isomorphisms between cyclotomic Hecke algebras and Khovanov-Lauda-Rouquier (KLR) algebras (of type A), see [2]. Since the KLR algebras are $\mathbb{Z}$-graded, the various Hecke algebras become $\mathbb{Z}$-graded in this way as well. On the other hand, $b_n(q, m)$ is known to be a quotient of the Hecke algebra $\mathcal{H}_n(q, Q)$ of type B, indeed it is also sometimes referred to as the Temperley-Lieb algebra of type B. But $\mathcal{H}_n(q, Q)$ is also the cyclotomic Hecke algebra of type $G(2, 1, n)$ and our basic idea is now to exploit the result from [2] on this quotient construction.

A big step towards our goal is taken already in section 3 of our paper, where we show that the ideal $\mathcal{J}_n \subset \mathcal{H}_n(q, Q)$, defining $b_n(q, m)$, is graded, thus making $b_n(q, m)$ a $\mathbb{Z}$-graded algebra. This result relies on a realization of $\mathcal{J}_n$ due to P. Martin and D. Woodcock in [14], in terms of certain explicitly given idempotents that turn our to be well behaved with respect to the KLR-relations.

On the other hand, this does not immediately imply a $\mathbb{Z}$-grading on the standard modules for $b_n(q, m)$ and indeed a major part of our paper is dedicated to this point. An important ingredient to this comes from the recent paper by Hu and Mathas, [11], that introduces the concept of a graded cellular algebra and shows that the cyclotomic Hecke algebras are graded cellular with respect to the $\mathbb{Z}$-grading given by Brundan and Kleshchev's work. We then achieve our goal in the sections 3-5 by showing that $b_n(q, m)$ is a graded cellular algebra.

A main difficulty in applying [11], is due to the fact that the cell structure on $\mathcal{H}_n(q, Q)$ considered in [11] is related to the dominance order on bipartitions, which is known to be incompatible with the natural order for the category of $b_n(q, m)$-modules, see [20] and [21]. We overcome this problem by showing that $b_n(q, m)$ is an algebra endowed with a family of Jucys-Murphy elements, in the sense of Mathas [17], with respect to a natural order that we introduce in section 3. This involves delicate arguments involving the diagram basis for $b_n(q, m)$.

It should be mentioned that our results are also valid in the Temperley-Lieb algebra case where the relevant Hecke algebra $\mathcal{H}_n(q)$ this time is of type A, and even in this case our results seem to be new. On the other hand, in the Temperley-Lieb algebra case there is actually a simpler way to show that the ideal of $\mathcal{H}_n(q)$ defining $Tl_n(q)$ is graded. It is based on certain properties of Murphy's standard basis that were proved by M. Härterich in [10].

Let us sketch the layout of the paper. In the next section we introduce the various algebras that play a role in the paper. In the third section we show that the ideals defining the Temperley-Lieb algebra and the blob algebra are graded, which makes these algebras graded. In the following section we recall the diagrammatic realizations of the Temperley-Lieb algebra and the blob algebra. We here focus mostly on the blob algebra case. We introduce two ways of parametrizing the blob diagrams, one via standard bitableaux of one-line bipartitions, the other via walks on the Bratteli diagram. We also introduce an order relation $\succ$ on the blob diagrams. In the fifth section we show that the images in $b_n(m)$ of the Jucys-Murphy elements of $\mathcal{H}_n(q, Q)$ make the blob algebra into an algebra with a family of Jucys-Murphy elements, in the sense of Mathas. As we explain in the beginning of that section, this is quite surprising. We rely here on both combinatorial descriptions of the blob diagrams. In the sixth section we obtain our main results, showing that the Temperley-Lieb algebra and the blob are both graded cellular, and in the last section we give two examples illustrating our results.

2. Notation and setup.

In this section we fix the notation that is used throughout the paper. We introduce the algebras to be studied, the Temperley-Lieb algebra, the blob algebra, the corresponding Hecke and Khovanov-Lauda-Rouquier algebras and recall the relevant results from the literature involving them. The important diagrammatic realizations of the Temperley-Lieb algebra and the blob algebra are postponed to section 4.

Throughout the paper the ground field is complex field $\mathbb{C}$ although some of our results hold in greater generality. For $q \in \mathbb{C}^{\times}$ and an integer k we define

$$[k] = [k]_q := q^{k-1} + q^{k-3} + \ldots + q^{-k+1} \in \mathbb{C}$$

the usual Gaussian integer. All our algebras are associative and unital.

2.1. The Temperley-Lieb algebra, the blob algebra, the Hecke algebras.

Definition 2.1. Let $q \in \mathbb{C}^{\times}$. The Temperley-Lieb algebra $Tl_n(q)$ is the $\mathbb{C}$-algebra on the generators $U_1, ..., U_{n-1}$ subject to the relations

$$
\begin{aligned}
U_i^2 &= -[2]U_i & &\text{if } 1 \leq i \leq n-1 \\
U_i U_j U_i &= U_i & &\text{if } |i - j| = 1 \\
U_i U_j &= U_j U_i & &\text{if } |i - j| > 1.
\end{aligned}
$$

The main object of the paper is the blob algebra, introduced in [13] by P. Martin and H. Saleur as a generalization of the Temperley-Lieb algebra. Let y_e be an invertible element of $\mathbb{C}$.

Definition 2.2. The blob algebra $b_n = b_n(q, y_e)$ is the $\mathbb{C}$-algebra on the generators $e, U_1, ..., U_{n-1}$ subject to the relations

$$
\begin{aligned}
U_i^2 &= -[2]U_i & &\text{if } 1 \leq i \leq n-1 \\
U_i U_j U_i &= U_i & &\text{if } |i - j| = 1 \\
U_i U_j &= U_j U_i & &\text{if } |i - j| > 1 \\
U_1 e U_1 &= y_e U_1 \\
e^2 &= e \\
U_i e &= e U_i & &\text{if } 2 \leq i \leq n-1.
\end{aligned}
$$

Assume that $[m] \neq 0$. The parametrization of b_n through $y_e = \frac{-[m-1]}{[m]}$ includes the non-semisimple cases, see [15, Section 2]. Under this choice of y_e we denote $b_n(q, y_e)$ by $b_n(m)$ and replace e by the rescaled generator $U_0 := -[m]e$.

The Temperley-Lieb algebra and the blob algebra were introduced from motivations in statistical mechanics. An important feature, that we postpone to the next section, is that they both have diagrammatic realizations by planar diagrams.

We next define the related Hecke algebras.

Definition 2.3. Let $q \in \mathbb{C}$ and assume that $q \neq 0, 1$. The Hecke algebra $\mathcal{H}_n(q)$ of type A_{n-1} is the $\mathbb{C}$-algebra with generators $T_1, \ldots, T_{n-1}$, subject to the relations

$$
\begin{aligned}
(T_i - q)(T_i + 1) &= 0 & &\text{for } 1 \leq i \leq n-1 & &(2.1) \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & &\text{for } 1 \leq i \leq n-2 & &(2.2) \\
T_i T_j &= T_j T_i & &\text{for } |i - j| > 1 & &(2.3)
\end{aligned}
$$

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It follows easily from the relations that T_r is an invertible element in $\mathcal{H}_n(q)$, with $T_r^{-1} = q^{-1}(T_r - q + 1)$. We may then define elements $L_1, \ldots, L_n \in \mathcal{H}_n(q)$ by $L_1 := 1$ and recursively $L_{r+1} = q^{-1}T_rL_rT_r$ for all admissible r. They are the first examples of Jucys-Murphy elements that play an important role in our paper.

Definition 2.4. Let $q, \lambda_1, \lambda_2 \in \mathbb{C}$ and suppose that $q \neq 0, 1$. The cyclotomic Hecke algebra $\mathcal{H}_n(q; \lambda_1, \lambda_2)$ of type $G(2, 1, n)$ is the $\mathbb{C}$-algebra with generators $L_1, \ldots, L_n$, $T_1, \ldots, T_{n-1}$ and relations

$$
\begin{array}{ll}
(L_1 - \lambda_1)(L_1 - \lambda_2) = 0, & L_rL_s = L_sL_r, \\
(T_r + 1)(T_r - q) = 0, & T_rL_r = L_{r+1}(T_r - q + 1), \\
T_sT_{s+1}T_s = T_{s+1}T_sT_{s+1}, & \\
T_rL_s = L_sT_r, & \text{if } |r - s| > 1, \\
T_rT_s = T_sT_r, & \text{if } s \neq r, r + 1
\end{array}
$$

for all admissible r, s.

Once again, T_r is invertible with $T_r^{-1} = q^{-1}(T_r - q + 1)$. From this one gets that $L_{r+1} = q^{-1}T_rL_rT_r$. Moreover, it follows from the relations that $f(L_1, \ldots, L_n)$ is a central element of $\mathcal{H}_n(q; \lambda_1, \lambda_2)$ for $f(x_1, \ldots, x_n)$ a symmetric polynomial. These L_i are also called Jucys-Murphy elements.

We now explain certain relations between the algebras that we have defined.

Theorem 2.5. *The are surjections Φ_1 and Φ_2 given by*

$$
\begin{array}{llllll}
\Phi_1 : & \mathcal{H}_n(q^2) & \longrightarrow & Tl_n(q), & T_i & \mapsto & qU_i + q^2 \\
\Phi_2 : & \mathcal{H}_n(q^2) & \longrightarrow & Tl_n(q), & T_i & \mapsto & -qU_i - 1.
\end{array}
$$

The kernel of Φ_1 is the ideal generated by

$$
q^{-6}T_1T_2T_1 - q^{-4}T_1T_2 - q^{-4}T_2T_1 + q^{-2}T_1 + q^{-2}T_1 - 1
$$

and the kernel of Φ_2 is the ideal generated by

$$
T_1T_2T_1 + T_1T_2 + T_2T_1 + T_1 + T_1 + 1.
$$

Proof: This is well known. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

There are two, not obviously equivalent, ways to generalize this Theorem to the blob algebra case. One is given in [9], but for our purposes it is more convenient to work with the second one, that appears in [14]. Set $Q := q^m$ and define $\mathcal{H}_n(m) = \mathcal{H}_n(q^2; Q, Q^{-1})$. Assume

$$
q^4 \neq 1, \qquad Q \neq Q^{-1}, \qquad Q \neq q^2Q^{-1}, \qquad Q^{-1} \neq q^2Q. \qquad (2.4)
$$

With the above conditions, one can define elements $e^{-1}, e^{-2} \in \mathcal{H}_2(m)$ by the formulas

$$
e^{-1} = \frac{(T_1 - q^2)(L_1 - Q^{-1})(L_2 - Q^{-1})}{(1 + q^2)(Q - Q^{-1})(Q^{-1} - q^{-2}Q)}
$$

$$
e^{-2} = \frac{(T_1 - q^2)(L_1 - Q)(L_2 - Q)}{(1 + q^2)(Q^{-1} - Q)(Q - q^{-2}Q^{-1})}.
$$

The factors of e^{-1} and e^{-2} commute with each other. Using this and $L_2 = q^{-2}T_1 L_1 T_1$, one finds that they verify the following equations

$$(T_1 + 1)e^{-1} = 0, \qquad\qquad (T_1 + 1)e^{-2} = 0, \qquad (2.5)$$

$$(L_1 - Q)e^{-1} = 0, \qquad\qquad (L_1 - Q^{-1})e^{-2} = 0, \qquad (2.6)$$

$$(L_2 - Qq^{-2})e^{-1} = 0, \qquad (L_2 - Q^{-1}q^{-2})e^{-2} = 0 \qquad (2.7)$$

and from this it follows that e^{-1} and e^{-2} are idempotents associated with irreducible representations of $\mathcal{H}_2(m)$ of dimension one. Note that e^{-1} and e^{-2} are the unique idempotents satisfying (2.5) and (2.6). They are denoted e_2^{-1} and e_2^{-2} in [14]. For all n there is a canonical embedding $\mathcal{H}_n(m) \hookrightarrow \mathcal{H}_{n+1}(m)$. Using it repeatedly we consider e^{-1} and e^{-2} as elements of $\mathcal{H}_n(m)$ and denote by $\mathcal{J}_n$ the ideal of $\mathcal{H}_n(m)$ generated by them.

Theorem 2.6. *The map Φ given by*

$$\begin{array}{rccl} \Phi : & \mathcal{H}_n(m) & \longrightarrow & b_n(m) \\ & T_i - q^2 & \mapsto & qU_i \\ & L_1 - q^m & \mapsto & (q - q^{-1})U_0 \end{array}$$

induces a $\mathbb{C}$-algebra isomorphism between $\mathcal{H}_n(m)/\mathcal{J}_n$ and $b_n(m)$.

Proof: See [14, Proposition 4.2]. □

We would like to have an integral version of the last result, but want also to avoid those choices of the parameters that correspond to the conditions (2.4). This can for example be achieved by localizing $\mathbb{C}[q, q^{-1}, Q, Q^{-1}]$ conveniently. To be precise, we choose for R the localization of the Laurent polynomial ring $\mathbb{C}[q, q^{-1}, Q, Q^{-1}]$ at S, defined as the multiplicatively closed subset of $\mathbb{C}[q, q^{-1}, Q, Q^{-1}]$ generated by the polynomials $1, q^4 - 1, Q - Q^{-1}, Q - Q^{-1}q^2$ and $Q^{-1} - Qq^2$. For integers e and m we denote by $\mathfrak{m}$ the ideal $\langle q - e^{2\pi i/e}, Q - q^m \rangle$ of R. Then we have that either $\mathfrak{m} = R$ or else $\mathfrak{m}$ is a maximal ideal in R. In the last case we define $\mathcal{O} := R_{\mathfrak{m}}$ and get that $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\mathfrak{m}$, quotient field $K := \mathbb{C}(q, Q)$ and residue field $\mathcal{O}/\mathfrak{m} = \mathbb{C}$ containing the e'th root of unity q.

Throughout the paper we assume that $\mathcal{O}, K$ and $\mathbb{C}$ are chosen as above. Thus, $q \in \mathbb{C}$ is an e'th root of unity. For technical reasons we moreover assume that e is odd.

We now define $b_n^{\mathcal{O}}(m)$ as the $\mathcal{O}$-algebra on generators $e, U_1, ..., U_{n-1}$ subject to the same relations as for b_n. Then $b_n^{\mathcal{O}}(m)$ is free over $\mathcal{O}$ as can be seen using the results of the appendix of [6], note that they are valid over any commutative ring. The rational blob algebra $b_n^K(m)$ is defined the same way, and we have base changes isomorphisms $b_n^{\mathcal{O}}(m) \otimes_{\mathcal{O}} \mathbb{C} = b_n(m)$ and $b_n^{\mathcal{O}}(m) \otimes_{\mathcal{O}} K = b_n^K(m)$. Finally we define $\mathcal{H}_n^{\mathcal{O}}(m)$ as the $\mathcal{O}$-algebra on generators $L_1, \ldots, L_n, T_1, \ldots, T_{n-1}$ subject to the same relations as for $\mathcal{H}_n(m)$, but using parameters $\lambda_1 = Q$ and $\lambda_2 = Q^{-1}$. Similarly, we define $\mathcal{H}_n^K(m)$ and we have base changes isomorphisms as above.

Theorem 2.7. *There is a surjection $\Phi : \mathcal{H}_n^{\mathcal{O}}(m) \longrightarrow b_n^{\mathcal{O}}(m)$.*

Proof: The argument given in [14, Proposition 4.2] involves verification of blob algebra relations and therefore gives a surjection $\mathcal{H}_n^{\mathcal{O}}(m) \longrightarrow b_n^{\mathcal{O}}(m)$, as claimed. □

2.2. The Khovanov-Lauda-Rouquier algebra.

Recall that the quantum characteristic of an element q of a field F is the smallest positive integer m such that $1 + q + \ldots + q^{m-1} = 0$, setting $m = 0$ if no such integer exists. With our choice of $q \in \mathbb{C}$ the quantum characteristic is e. We set $I = \mathbb{Z}/e\mathbb{Z}$ and refer to I^n as the residue sequences of length n.

In the following $\mathcal{H}$ refers to either $\mathcal{H}_n(q^2)$ or $\mathcal{H}_n(m)$ (with $q \in \mathbb{C}$ chosen as above). Let M be a finite dimensional $\mathcal{H}$-module. By [12, Lemma 7.1.2] the eigenvalues of each L_r on M are of the form q^{2i} for $i \in I$. So M decomposes as the direct sum $M = \bigoplus_{\boldsymbol{i} \in I^n} M_{\boldsymbol{i}}$ of its generalized weight spaces

$$M_{\boldsymbol{i}} := \{v \in M \ \mid \ (L_r - q^{2i_r})^k v = 0 \text{ for } r = 1, \ldots, n \text{ and } k \gg 0\}.$$

In particular, taking M to be the regular left module $\mathcal{H}$, we obtain a system $\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\}$ of mutually orthogonal idempotents in $\mathcal{H}$ such that $e(\boldsymbol{i})M = M_{\boldsymbol{i}}$ for each M as above.

We can now define nilpotent elements $y_1, \ldots, y_n \in \mathcal{H}$ via the formula

$$y_r = \sum_{\boldsymbol{i} \in I^n} (1 - q^{-2i_r} L_r) e(\boldsymbol{i}). \tag{2.8}$$

For $1 \le r < n$ and $\boldsymbol{i} \in I^n$, Brundan and Kleshchev define in [2] certain formal power series, $P_r(\boldsymbol{i}), Q_r(\boldsymbol{i}) \in \mathbb{C}[[y_r, y_{r+1}]]$, such that $Q_r(\boldsymbol{i})$ has non-zero constant term, see [2, (4.27) and (4.36)] for the explicit formulas. Since each y_r is nilpotent in $\mathcal{H}$, we can consider $P_r(\boldsymbol{i})$ and $Q_r(\boldsymbol{i})$ as elements of $\mathcal{H}$, with $Q_r(\boldsymbol{i})$ invertible. We then set

$$\psi_r = \sum_{\boldsymbol{i} \in I^n} (T_r + P_r(\boldsymbol{i})) Q_r(\mathbf{i})^{-1} e(\boldsymbol{i}). \tag{2.9}$$

The main theorem in [2] gives a presentation of $\mathcal{H}$ in terms of the elements

$$\{\psi_1, \cdots, \psi_{n-1}\} \cup \{y_1, \cdots, y_n\} \cup \{e(\mathbf{i}) \mid \mathbf{i} \in I^n\}$$

and a series of relations between them that we describe shortly. An important point of these relations is that they are homogeneous with respect to a nontrivial $\mathbb{Z}$-grading on $\mathcal{H}$. To describe the $\mathbb{Z}$-grading it is convenient to introduce the matrix $(a_{ij})_{i,j \in I}$, given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 0 & \text{if } i \ne j \pm 1 \\ -1 & \text{if } i = j \pm 1. \end{cases}$$

With this at hand, we are now able to state [2, Main Theorem]. The Theorem holds in greater generality than shown here, namely for all cyclotomic Hecke algebras, including the degenerate algebras, but for our purpose the following version is enough.

Theorem 2.8. *The algebra $\mathcal{H}$ is isomorphic to a cyclotomic Khovanov-Lauda-Rouquier algebra of type A. To be precise, it is isomorphic to the $\mathbb{C}$-algebra generated by*

$$\{\psi_1, \cdots, \psi_{n-1}\} \cup \{y_1, \cdots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\}$$

subject to the following relations for $\boldsymbol{i}, \boldsymbol{j} \in I^n$ and all admissible r, s

$$y_1 e(\boldsymbol{i}) = 0 \ \textit{if} \ i_1 = \begin{cases} \pm k \mod e & \textit{if } \mathcal{H} = \mathcal{H}_n(m) \\ 0 \mod e & \textit{if } \mathcal{H} = \mathcal{H}_n(q^2) \end{cases} \tag{2.10}$$

$$e(\boldsymbol{i}) = 0 \ \textit{if} \ i_1 \neq \begin{cases} \pm k \mod e & \textit{if } \mathcal{H} = \mathcal{H}_n(m) \\ 0 \mod e & \textit{if } \mathcal{H} = \mathcal{H}_n(q^2) \end{cases} \tag{2.11}$$

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}), \tag{2.12}$$

$$\sum_{\boldsymbol{i} \in I^n} e(\boldsymbol{i}) = 1, \tag{2.13}$$

$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i}) y_r, \tag{2.14}$$

$$\psi_r e(\boldsymbol{i}) = e(s_r \boldsymbol{i}) \psi_r, \tag{2.15}$$

$$y_r y_s = y_s y_r, \tag{2.16}$$

$$\psi_r y_s = y_s \psi_r, \qquad \qquad \textit{if } s \neq r, r+1 \tag{2.17}$$

$$\psi_r \psi_s = \psi_s \psi_r, \qquad \qquad \textit{if } |s - r| > 1 \tag{2.18}$$

$$\psi_r y_{r+1} e(\boldsymbol{i}) = \begin{cases} (y_r \psi_r + 1) e(\boldsymbol{i}) & \textit{if } i_r = i_{r+1} \\ y_r \psi_r e(\boldsymbol{i}) & \textit{if } i_r \neq i_{r+1} \end{cases} \tag{2.19}$$

$$y_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_r y_r + 1) e(\boldsymbol{i}) & \textit{if } i_r = i_{r+1} \\ \psi_r y_r e(\boldsymbol{i}) & \textit{if } i_r \neq i_{r+1} \end{cases} \tag{2.20}$$

$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \textit{if } i_r = i_{r+1} \\ e(\boldsymbol{i}) & \textit{if } i_r \neq i_{r+1} \pm 1 \\ (y_{r+1} - y_r) e(\boldsymbol{i}) & \textit{if } i_{r+1} = i_r + 1 \\ (y_r - y_{r+1}) e(\boldsymbol{i}) & \textit{if } i_{r+1} = i_r - 1 \end{cases} \tag{2.21}$$

$$\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\boldsymbol{i}) & \textit{if } i_{r+2} = i_r = i_{r+1} - 1 \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\boldsymbol{i}) & \textit{if } i_{r+2} = i_r = i_{r+1} + 1 \\ (\psi_{r+1} \psi_r \psi_{r+1}) e(\boldsymbol{i}) & \textit{otherwise} \end{cases} \tag{2.22}$$

where $s_r := (r, r+1)$ is the simple transposition acting in I^n by permutation of the coordinates $r, r+1$ and $k \in \mathbb{Z}$ such that $2k \equiv m \mod e$. The isomorphism maps each of the generators to the element of $\mathcal{H}$ that has the same name. The conditions

$$\deg e(\boldsymbol{i}) = 0, \qquad \deg y_r = 2, \qquad \deg \psi_s e(\boldsymbol{i}) = -a_{i_s, i_{s+1}}$$

for $1 \leq r \leq n$, $1 \leq s \leq n-1$ and $\boldsymbol{i} \in I^n$ define a unique $\mathbb{Z}$-grading on $\mathcal{H}$ with degree function deg.

Following [11] we shall refer to the $e(\boldsymbol{i})$ as the KLR-idempotents. In the following, all statements involving a grading on $\mathcal{H}$ refer to the above Theorem. Note that although the elements L_r and T_r are not homogeneous in $\mathcal{H}$, they can be expressed in terms of homogeneous generators in the following way, see equations (4.42) and (4.43) of [2]:

$$L_r = \sum_{\boldsymbol{i} \in I^n} q^{2i_r} (1 - y_r) e(\boldsymbol{i}) \tag{2.23}$$

$$T_r = \sum_{\boldsymbol{i} \in I^n} (\psi_r Q_r(\boldsymbol{i}) - P_r(\boldsymbol{i})) e(\boldsymbol{i}). \tag{2.24}$$

2.3. Partitions and tableaux.

We finish this introductory section by recalling some basic combinatorial notions related to the symmetric group. Whenever we work with a partially ordered set X with order relation say $\geq$, we write $x_1 > x_2$, that is omit the lower line of the order symbol, if $x_1, x_2 \in X$ satisfy $x_1 \geq x_2$ and $x_1 \neq x_2$. Let n be a positive integer. A(n integer) partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $|\lambda| := \sum_i \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$. The Young diagram of λ is the set

$$[\lambda] = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \lambda_i \text{ and } i \geq 1\}.$$

The elements of it are called nodes or entries. It is useful to think of $[\lambda]$ as an array of boxes in the plane, with the indices following matrix conventions. Thus the box with label (i,j) belongs to the i'th row and j'th column. If λ is a partition of n we denote by λ' the partition of n obtained from λ by interchanging its rows and columns. A two-column partition of n is a partition λ of n such that $\lambda_i \leq 2$ for all $i \geq 1$. The set of all partitions of n is denoted $\text{Par}(n)$ and the set of two-column partitions of n is denoted by $\text{Par}_2(n)$. A λ-tableau is a bijection $\tau : [\lambda] \to \{1, \ldots, n\}$. We say that τ has shape λ and write $\text{Shape}(\tau) = \lambda$. We think of it as a labeling of the diagram of λ, and in this way we can talk of the rows and columns of a tableau. We say that τ is row (resp. column) standard if the entries of τ increase from left (resp. top) to right (resp. bottom) in each row (resp. column). τ is standard if it is row standard and column standard. The set of all standard λ-tableau is denoted by $\text{Std}(\lambda)$ and the union of all $\text{Std}(\lambda)$ is denoted $\text{Std}(n)$.

Assume that $\lambda, \mu \in \text{Par}(n)$. We say that λ dominates μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all $j \geq 1$. Then $\text{Par}(n)$ becomes a partially ordered set via $\trianglerighteq$. It can be extended to $\text{Std}(n)$ as follows. For $\sigma, \tau \in \text{Std}(n)$, we say that σ dominates τ and write $\sigma \trianglerighteq \tau$ if $Shape(\sigma_{|_k}) \trianglerighteq \text{Shape}(\tau_{|_k})$, for $k = 1, \ldots, n$, where $\sigma_{|_k}$ and $\tau_{|_k}$ are the tableaux obtained from σ and τ by removing the entries greater than k.

Let τ^λ be the unique standard λ-tableau such that $\tau^\lambda \trianglerighteq \tau$ for all $\tau \in \text{Std}(\lambda)$. In τ^λ the numbers $1, 2, \ldots, n$ are filled in increasingly along the rows from top to bottom. The symmetric group $\mathfrak{S}_n$ acts on the left on the set of λ-tableaux permuting the entries. For $\tau \in \text{Std}(\lambda)$, we denote by $d(\tau)$ the permutation of $\mathfrak{S}_n$ that satisfies $\tau = d(\tau)\tau^\lambda$.

3. GRADING THE TEMPERLEY-LIEB ALGEBRA AND THE BLOB ALGEBRA.

In this section we show that the Temperley-Lieb algebra $Tl_n(q)$ and the blob algebra $b_n(m)$ are $\mathbb{Z}$-graded algebras. We do this by proving that the kernels of the surjections given in Theorem 2.6 and Theorem 2.5 are graded ideals. In the $Tl_n(q)$-case we rely on certain properties of Murphy's standard basis that are proved in [10]. These properties are missing in the $b_n(m)$-case and so our argument is somewhat different in that case. Let $A = \oplus_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded ring with homogeneous parts A_n. Recall that $I \subset A$ is called a graded (homogeneous) ideal of A if it is an ideal and if $I = \oplus_{n \in \mathbb{Z}} I_n$ where $I_n := A_n \cap I$. If I is a graded ideal of A then the quotient A/I becomes a $\mathbb{Z}$-graded ring as well, with homogeneous parts A_n/I_n. We need the following Theorem.

Theorem 3.1. *Let A be a $\mathbb{Z}$-graded algebra. Assume that I is an ideal of A that is generated by homogeneous elements. Then I is graded.*

Proof: See [5, Theorem 1.3]. $\qquad\square$

3.1. Grading $Tl_n(q)$.

Let us briefly recall Murphy's standard basis for the Hecke algebra $\mathcal{H}_n(q^2)$. For $w = s_{i_1} \ldots s_{i_k}$ a reduced expression of $w \in \mathfrak{S}_n$ we define $T_w := T_{i_1} \ldots T_{i_k}$. Then $\{T_w | w \in \mathfrak{S}_n\}$ is a basis for $\mathcal{H}_n(q)$. For $\lambda \in \mathrm{Par}(n)$ we let $\mathfrak{S}_\lambda \leq \mathfrak{S}_n$ denote the row stabilizer of τ^λ under the left action of $\mathfrak{S}_n$ on tableaux and define

$$x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w$$

We let $*$ denote the anti-automorphism of $\mathcal{H}_n$ determined by $T_i^* = T_i$ for all $1 \leq i < n$ and define for $\sigma, \tau \in \mathrm{Std}(\lambda)$

$$x_{\tau\sigma} = T_{d(\tau)}^* x_\lambda T_{d(\sigma)}.$$

Then $\{x_{\tau\sigma}\}$, with τ and σ running over standard tableaux of the same shape, is Murphy's standard basis for $\mathcal{H}_n(q^2)$, see [19, Theorem 4.17].

We set $\mathcal{I}_n := \ker \Phi_2$ where $\Phi_2 : \mathcal{H}_n(q^2) \longrightarrow Tl_n(q)$ is the second surjection given in Theorem 2.5. Then $\mathcal{I}_n$ is an ideal of $\mathcal{H}_n(q^2)$ and we have $\mathcal{H}_n(q^2)/\mathcal{I}_n = Tl_n(q)$. We can now state our first Theorem.

Theorem 3.2. *$\mathcal{I}_n$ is a graded ideal of $\mathcal{H}_n(q^2)$. Hence $Tl_n(q)$ is a $\mathbb{Z}$-graded algebra, with the grading induced from the one on $\mathcal{H}_n(q^2)$, via Theorem 2.8.*

Proof: We first note that by the results of Härterich, [10, Theorem 4], we know that $\mathcal{I}_n$ is spanned (over $\mathbb{C}$!) by those $\{x_{\tau\sigma}\}$ for which the underlying shape has strictly more than two columns, that is $\mathrm{Shape}(\tau), \mathrm{Shape}(\sigma) \notin \mathrm{Par}_2(n)$. In other words, $\{x_{\tau\sigma} \,|\, \sigma, \tau \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(n) \setminus \mathrm{Par}_2(n)\}$ is a basis for $\mathcal{I}_n$.

On the other hand, in [11] J. Hu and A. Mathas construct a basis $\{\psi_{\tau\sigma}\}$ for $\mathcal{H}_n(q^2)$, such that each $\psi_{\tau\sigma}$ is a homogeneous element of $\mathcal{H}_n(q^2)$; here (τ, σ) is running over the same set as for the standard basis. They furthermore show in [11, Lemma 5.4] that for each pair (τ, σ) like this, there is a non-zero scalar $c \in \mathbb{C}$ such that

$$\psi_{\tau\sigma} = cx_{\tau\sigma} + \sum_{(\upsilon,\varsigma) \rhd (\tau,\sigma)} r_{\upsilon\varsigma} x_{\upsilon\varsigma} \tag{3.1}$$

where $r_{\upsilon\varsigma} \in \mathbb{C}$ and where $(\upsilon, \varsigma) \rhd (\tau, \sigma)$ means that $\upsilon \unrhd \tau$, $\varsigma \unrhd \sigma$, and $(\upsilon, \varsigma) \neq (\tau, \sigma)$. But this shows that also the $\{\psi_{\tau\sigma}\}$ such that $\mathrm{Shape}(\tau), \mathrm{Shape}(\sigma) \notin \mathrm{Par}_2(n)$, are a basis for $\mathcal{I}_n$. From this we get, via Theorem 3.1 that $\mathcal{I}_n$ is a graded ideal as claimed. $\qquad\square$

Remark 3.3. There is a version of the Theorem involving the homomorphism Φ_1. For this, in the proof one should replace $\{\psi_{\tau\sigma}\}$ by the dual basis $\{\psi'_{\tau\sigma}\}$ of [11].

Remark 3.4. In spite of the important role of the Temperley-Lieb algebra in recent categorification theory, see eg. [22], the above graded structure has not been mentioned before in the literature, to the best of our knowledge. Our grading is also not immediately related to the supergrading used in [23].

3.2. Grading $b_n(m)$.

Let us now turn to the blob algebra. In order to treat that case we need the following Theorem. Note that there is always $k \in \mathbb{Z}$ satisfying the condition of the Theorem, since e is assumed to be odd.

Theorem 3.5. *Let $k \in \mathbb{Z}$ such that $2k \equiv m \mod e$. Then, the elements $e^{-1}, e^{-2} \in \mathcal{H}_n(m)$ are homogeneous of degree zero. More precisely, they can be written as a sum of homogeneous elements of degree zero as follows*

$$e^{-1} = \sum_{\boldsymbol{i}} e(\boldsymbol{i}) \qquad\qquad e^{-2} = \sum_{\boldsymbol{j}} e(\boldsymbol{j})$$

where the left sum runs over all $\boldsymbol{i} \in I^n$ such that $i_1 = k$ and $i_2 = k - 1$, and the right sum runs over all $\boldsymbol{j} \in I^n$ such that $j_1 = -k$ and $j_2 = -k - 1$.

Proof: We only prove the result for e^{-1}, the result for e^{-2} is proved similarly.

In [4, Section 4.4], Brundan, Kleshchev and Wang note that under the embedding $\mathcal{H}_n(m) \hookrightarrow \mathcal{H}_{n+1}(m)$ one has $e(\boldsymbol{i}) \mapsto \sum_{i \in I} e(\boldsymbol{i}, i)$, and so it is enough to prove the case $n = 2$, that is that $e^{-1} = e(k, k-1)$ holds. Using the uniqueness statement for e^{-1}, in order to prove this, it is enough to show that $e(k, k-1)$ verifies the equations (2.5) and (2.6), since it is clearly an idempotent.

Note first that $y_1 = 0$ as it follows by combining the relations (2.10), (2.11) and (2.13). Put now $\boldsymbol{j} = (k, k-1)$. Multiplying (2.23) by $e(\boldsymbol{j})$ for $n = 2$ and $r = 1$, we get $L_1 e(\boldsymbol{j}) = q^{2k} e(\boldsymbol{j})$, or equivalently $L_1 e(\boldsymbol{j}) = q^m e(\boldsymbol{j})$. Hence (2.6) holds.

To show (2.5) we first recall from [11, Lemma 4.1(c)] that in general $e(\boldsymbol{i}) \neq 0$ iff $\boldsymbol{i} \in I^n$ is a residue sequence coming from a standard bi-tableau of a bipartition of n. Combining this fact with the standing conditions on q given in (2.4), we deduce $e(s_1 \boldsymbol{j}) = 0$ and hence $\psi_1 e(\boldsymbol{j}) = 0$ by (2.15). Multiplying this equation on the left by ψ_1 and using (2.21) we obtain $y_2 e(\boldsymbol{j}) = 0$.

Now, recall that by definition $P_1(\boldsymbol{j})$ and $Q_1(\boldsymbol{j})$ are power series in y_1 and y_2. Furthermore, in this particular case we have that the constant coefficient of $P_1(\boldsymbol{j})$ is 1 and so (2.24) gives $(T_1 + 1)e(\boldsymbol{j}) = 0$ as needed. $\qquad\square$

We are now in position to establish the main objective of this section, namely to provide a graded structure on $b_n(m)$. In the forthcoming Section 6, we refine this graded structure on $b_n(m)$ to a graded cellular basis structure.

Corollary 3.6. *The kernel of the surjection $\Phi : \mathcal{H}_n(m) \longrightarrow b_n(m)$ from Theorem 2.6 is a graded ideal. Hence, the algebra $b_n(m)$ has a presentation with generators*

$$\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\}$$

subject to the same relations as in Theorem 2.8, with the additional relation

$$e(\boldsymbol{i}) = 0$$

whenever $i_1 = k$ and $i_2 = k - 1$, or $i_1 = -k$ and $i_2 = -k - 1$. These relations are homogeneous with respect to the degree function defined in Definition 2.8. Therefore, $b_n(m)$ can be provided with the structure of a $\mathbb{Z}$-graded algebra such that Φ is a homogeneous homomorphism.

Proof: The result follows by direct application of Theorems 2.6, 3.5 and 3.1. $\qquad\square$

4. DIAGRAMS ALGEBRAS AND COMBINATORICS OF TABLEAUX.

In this section, we introduce diagram bases for the Temperley-Lieb algebra and the blob algebra. We index the diagram basis of the Temperley-Lieb algebra by pairs of two-column standard tableaux. Similarly, we index the diagram basis of the blob algebra by pairs of one-line standard bitableaux.

4.1. Diagram basis for $Tl_n(q)$.

We first recall the diagrammatic realization of the Temperley-Lieb algebra $Tl_n(q)$, first given by L. Kauffman, in which the basis elements are drawn as "(n,n)-bridges" or simply "Temperley-Lieb diagrams". An (n,n)-bridge consists of n vertices, also called points or nodes, on each of two parallel edges, the "top" resp. "bottom" lines, that are joined pairwise by n non-intersecting lines between the two lines. Figure 1 shows two examples.
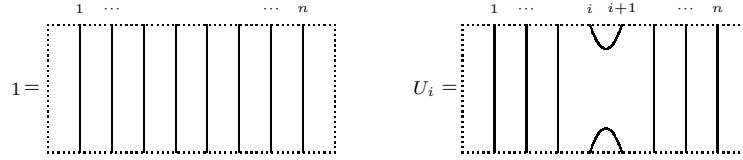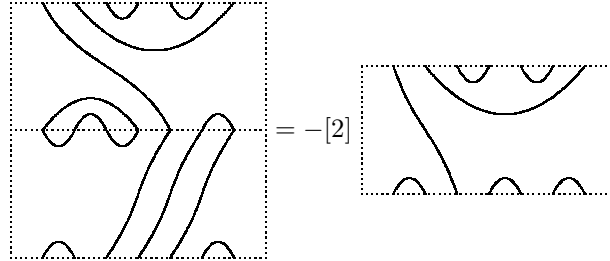

Figure 1: Diagrammatic generators

The set of all (n,n)-bridges is denoted by $\mathbb{T}(n)$. We define a multiplication on $\mathbb{CT}(n)$ by identifying the bottom of the first diagram with the top of the second, and replacing every closed loop that may arise by a factor $-[2]$ (see Figure 2).

Figure 2: Composition in $Tl_7(q)$

With this definition $\mathbb{CT}(n)$ becomes a $\mathbb{C}$-algebra where the one-element is the diagram denoted by 1 in Figure 1 . The diagrammatic realization of the Temperley-Lieb algebra refers to the isomorphism of $\mathbb{C}$-algebras $f : Tl_n \to \mathbb{CT}(n)$, given by $f(U_i) = U_i$ where the second U_i is the diagram of Figure 1.

Let us now recall the bijection between (n,n)-bridges and pairs of two-column standard tableaux of the same shape. Let β be an element of $\mathbb{T}(n)$. We say that a line of β is vertical if it travels from top to bottom, otherwise we say that it is horizontal. Suppose now that β has exactly v vertical lines and set $h = \frac{n-v}{2}$. The associated pair of standard $(h+v, h)'$-tableaux $(\tau_{top}(\beta), \tau_{bot}(\beta))$ is then given by the following rules:

1. k is in the second column of $\tau_{top}(\beta)$ $(\tau_{bot}(\beta))$ if and only if the k-th point is a right endpoint of a horizontal line in the top (bottom) edge.

2. the numbers increase along the columns of $\tau_{top}(\beta)$ and $\tau_{bot}(\beta)$.

For $\lambda \in \mathrm{Par}_2(n)$ and $\sigma, \tau \in \mathrm{Std}(\lambda)$, we denote by $\beta_{\sigma\tau}$ the unique (n,n)-bridge such that $\tau_{top}(\beta_{\sigma\tau}) = \sigma$ and $\tau_{bot}(\beta_{\sigma\tau}) = \tau$.

4.2. Bipartitions, bitableaux and diagrammatic realization of b_n.

We aim at generalizing the above results to the case of the blob algebra. For this we first recall the concepts of bipartitions and bitableaux. We provide them with structures of partially ordered sets, in a non-conventional way.

A bipartition of n is a pair $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of usual (integer) partitions such that $n = |\lambda^{(1)}| + |\lambda^{(2)}|$. By the Young diagram of a bipartition $\boldsymbol{\lambda}$ we mean the set

$$[\boldsymbol{\lambda}] = \{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \{1,2\} \mid 1 \leq j \leq \lambda_i^{(k)}\}.$$

Its elements are called entries or nodes. We can visualize $[\boldsymbol{\lambda}]$ as a pair of usual Young diagrams called the components of $[\boldsymbol{\lambda}]$. Thus for $l = 1, 2$, the l'th component of $[\boldsymbol{\lambda}]$ is $\{(i,j,k) \in [\boldsymbol{\lambda}] \mid k = l\}$. A one-line bipartition of n is a bipartition $\boldsymbol{\lambda}$ of n such that $\lambda_i^{(k)} = 0$ for all $i \geq 2$ and $k = 1, 2$. The set of all one-line bipartitions of n is denoted $\mathrm{Bip}_1(n)$. For $\boldsymbol{\lambda}$ a bipartition, a $\boldsymbol{\lambda}$-bitableau is a bijection $\mathfrak{t} : [\boldsymbol{\lambda}] \to \{1, \ldots, n\}$. We say that $\mathfrak{t}$ has shape $\boldsymbol{\lambda}$ and write $\mathrm{Shape}(\mathfrak{t}) = \boldsymbol{\lambda}$. A $\boldsymbol{\lambda}$-bitableau $\mathfrak{t}$ is called standard if the entries of $\mathfrak{t}$ increase from left to right in each component. The set of all standard $\boldsymbol{\lambda}$-bitableaux is denoted by $\mathrm{Std}(\boldsymbol{\lambda})$ and the union $\bigcup_{\boldsymbol{\lambda}} \mathrm{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda}$ running over all bipartitions of n is denoted by $\mathrm{Std}(\boldsymbol{n})$.

There are several ways of endowing $\mathrm{Bip}_1(n)$ with an order structure, the most well known being dominance order, but we shall need a different order on $\mathrm{Bip}_1(n)$ that we now explain. Let Λ_n be the set $\{-n, -n+2, \ldots, n-2, n\}$. Then the following definition makes Λ_n into a totally ordered set with order relation $\succ$.

Definition 4.1. *Suppose $\lambda, \mu \in \Lambda_n$. We then define $\mu \succeq \lambda$ if either $|\mu| < |\lambda|$, or if $|\mu| = |\lambda|$ and $\mu \leq \lambda$.*

On the other hand, the map f given by

$$f : \mathrm{Bip}_1(n) \to \Lambda_n, \ ((a),(b)) \to a - b$$

is a bijection and so we can define a total order $\succeq$ on $\mathrm{Bip}_1(n)$ as follows.

Definition 4.2. *Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in Bip_1(n)$. Then we define $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$ iff $f(\boldsymbol{\lambda}) \succeq f(\boldsymbol{\mu})$.*

For $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ let $\mathfrak{t}_{|k}$ be the tableau obtained from $\mathfrak{t}$ by removing the entries greater than k. We extend the order $\succeq$ to the set of all $\boldsymbol{\lambda}$-standard bitableaux as follows.

Definition 4.3. *Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. We define $\mathfrak{s} \succeq \mathfrak{t}$ if $\mathrm{Shape}(\mathfrak{s}_{|k}) \succeq \mathrm{Shape}(\mathfrak{t}_{|k})$ for all $k = 1, \ldots, n$.*

Note that this is only a partial order on $\mathrm{Std}(\boldsymbol{\lambda})$. Let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the unique standard $\boldsymbol{\lambda}$-bitableau such that $\mathfrak{t}^{\boldsymbol{\lambda}} \succeq \mathfrak{t}$ for all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. For $\boldsymbol{\lambda} = (a,b)$, set $m = \min\{a, b\}$. Then in $\mathfrak{t}^{\boldsymbol{\lambda}}$ the numbers $1, 2, \ldots, n$ are located increasingly along the rows according to the following rules:

1. even numbers less than or equal to $2m$ are placed in the first component.

2. odd numbers less than $2m$ are placed in the second component.

3. numbers greater than $2m$ are placed in the remaining boxes.

Definition 4.4. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Define a sequence of integers inductively by the rules $\mathfrak{t}(0) = 0$ and for $1 \leq j \leq n$

$$\mathfrak{t}(j) = \mathfrak{t}(j-1) \pm 1$$

where the $+$ $(-)$ sign is used if j is in the first (second) component of $\mathfrak{t}$.

Using this sequence we can now describe the order $\succeq$.

Lemma 4.5. *If* $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$*, then* $\mathfrak{s} \succeq \mathfrak{t}$ *if and only if* $|\mathfrak{s}(j)| \leq |\mathfrak{t}(j)|$*, for all* $1 \leq j \leq n$*, and if* $|\mathfrak{s}(j)| = |\mathfrak{t}(j)|$ *then* $\mathfrak{s}(j) \leq \mathfrak{t}(j)$*.*

Proof: Note that for all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \leq j \leq n$, we have $\mathfrak{t}(j) = f(\mathrm{Shape}(\mathfrak{t}|_j))$. Therefore, the result is a direct consequence of Definition 4.3. $\square$

As is the case for the Temperley-Lieb algebra, the blob algebra has a diagrammatic realization that we now explain. A "blob diagram on n points", or just a blob diagram when no confusion arises, is an (n, n)-bridge with possible decorations of "blobs" on certain of its lines. The blobs appear subject to the following conditions. Each line is decorated with at most one blob; no line to the right of the leftmost vertical line may be decorated; and to the left of it, only the outermost line in any nested formation of loop lines can be decorated. The set of blob diagrams on n points is denoted $\mathbb{B}(n)$. Similar to the Temperley-Lieb case, there is now a multiplication on $\mathbb{CB}(n)$, defined using a concatenation procedure. This may give rise to internal loops and multiple blobs on certain lines. We then impose the rules on the multiplication that any diagram with multiple blobs on one or several lines is considered equal to the same diagram with a single blob on those lines, and any internal loop is removed from the diagram multiplying by y_e, if the loop is decorated, otherwise by $-(q + q^{-1})$. The realization of $b_n(m)$ is now the isomorphism $f : b_n(m) \to \mathbb{CB}(n)$, mapping U_i and e to the diagrams U_i and e, given in Figure 1 and 3.
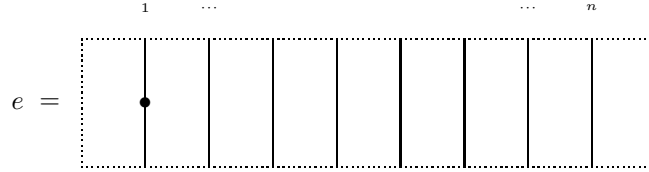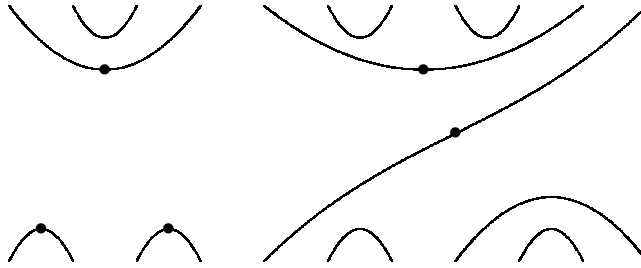

Figure 3: Blob generator e.

Figure 4: A diagram of b_{11}

Our next goal is to establish a bijection between the set of blob diagrams and the set of pairs of one-line standard bitableaux of same shape. Let m be a blob

diagram. Given a horizontal line l, in either edge, we put $l = (a, b)$ where a is the left endpoint and b is the right endpoint. Let $l_1 = (a_1, b_1)$ and $l_2 = (a_2, b_2)$ be horizontal lines on the same edge. We say that l_1 covers l_2 if $a_1 < a_2 < b_2 < b_1$. We also say that the leftmost vertical line (if any) covers all lines to the right of it. Now, we say that a node is covered if the line to which it belongs is decorated or the line to which it belongs is covered by a decorated line. If a node is not covered, we call it uncovered.

Definition 4.6. Let m be a blob diagram. Suppose that m has exactly v vertical lines and $h = \frac{n-v}{2}$ horizontal lines on each edge.

- If $v \geq 0$ and the leftmost vertical line is not decorated or there is no vertical lines then we associate to m a pair of $\boldsymbol{\lambda}$-bitableaux, $\mathfrak{t}_{top}(m)$ and $\mathfrak{t}_{bot}(m)$, with $\boldsymbol{\lambda} = ((h + v), (h))$ by the following rules

  1. k is in the second component of $\mathfrak{t}_{top}(m)$ $(\mathfrak{t}_{bot}(m))$ if and only if: either k is uncovered and it is the right endpoint of a horizontal line on the top (bottom) edge, or it is covered and it is the left endpoint of a horizontal line on the top (bottom) edge

  2. the numbers increase along rows.

- If $v > 0$ and the leftmost vertical line is decorated then we associate to m a pair of $\boldsymbol{\lambda}$-bitableaux, $\mathfrak{t}_{top}(m)$ and $\mathfrak{t}_{bot}(m)$, with $\boldsymbol{\lambda} = ((h), (h + v))$ by the following rules

  1. k is in the first component of $\mathfrak{t}_{top}(m)$ $(\mathfrak{t}_{bot}(m))$ if and only if: either it is uncovered and it is the left endpoint of a horizontal line on the top (bottom) edge or it is covered and it is the right endpoint of a horizontal line on the top (bottom) edge

  2. the numbers increase along rows.

We view these rules as a generalization of the bijection between $\mathbb{T}(n)$ and $\mathrm{Par}_2(n)$, with the two components of the bitableau replacing the two columns of the element of $\mathrm{Par}_2(n)$ and with the presence of a cover reversing the roles of left and right.

For $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, we let $m_{\mathfrak{s}\mathfrak{t}}$ denote the unique blob diagram such that $\mathfrak{t}_{top}(m_{\mathfrak{s}\mathfrak{t}}) = \mathfrak{s}$ and $\mathfrak{t}_{bot}(m_{\mathfrak{s}\mathfrak{t}}) = \mathfrak{t}$.

Remark 4.7. For all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \leq j \leq n$, we have

(i) If $\mathfrak{t}(j) < 0$ then the node k is covered in the top edge of $m_{\mathfrak{t}\mathfrak{t}^\lambda}$.

(ii) If the node k is covered in the top edge of $m_{\mathfrak{t}\mathfrak{t}^\lambda}$ then $\mathfrak{t}(j) \leq 0$.

Example 4.8. Let m be the diagram in Figure 4 then

$$\mathfrak{t}_{top}(m) = (\;\boxed{3\;|\;4\;|\;7\;|\;9\;|\;10}\;,\;\boxed{1\;|\;2\;|\;5\;|\;6\;|\;8\;|\;11}\;)$$

$$\mathfrak{t}_{bot}(m) = (\;\boxed{2\;|\;4\;|\;7\;|\;10\;|\;11}\;,\;\boxed{1\;|\;3\;|\;5\;|\;6\;|\;8\;|\;9}\;)$$

4.3. Walks on the Bratteli diagram.

We now explain how the Bratteli diagram for $b_n(m)$ provides a useful interpretation of the order $\succ$ on $\mathrm{Std}(\boldsymbol{\lambda})$. Let $\mathbb{B}^{top}(n)$ (resp. $\mathbb{B}^{bot}(n)$) denote the set of upper (lower) halves of blob diagrams. To be more precise, $\mathbb{B}^{top}(n)$ (resp. $\mathbb{B}^{bot}(n)$) consists of all blob diagrams on n points with the information on the bottom (top) points of the vertical lines omitted. Thus $\mathbb{B}^{top}(n)$ (resp. $\mathbb{B}^{bot}(n)$) is in bijection with $\mathrm{Std}(\boldsymbol{n})$ via $m \mapsto \mathfrak{t}_{top}(m)$ (resp. $m \mapsto \mathfrak{t}_{bot}(m)$) and so $\mathbb{B}^{top}(n)$ and $\mathbb{B}^{bot}(n)$ are in bijection with each other. On the diagrammatic level, the bijection can be visualized as a reflection through an appropriate horizontal axis.

Recall that the Bratteli diagram for $b_n(m)$ gives an enumeration of $\mathbb{B}^{top}(n)$ through a Pascal triangle pattern, see [14]. To be precise, for $\lambda \in \Lambda_n$ the Bratteli diagram associates with the point (λ, n) of the plane the set $\mathbb{B}^{top}(n, \lambda)$, defined as those diagrams from $\mathbb{B}^{top}(n)$ that have exactly $|\lambda|$ vertical lines, where the leftmost vertical line is decorated iff λ is negative. Set $b_{n,\lambda} := |\mathbb{B}^{top}(n, \lambda)|$ with the convention that $\mathbb{B}^{top}(n, \lambda) := \emptyset$ if $\lambda \notin \Lambda_n$. Then there is a bijection between $\mathbb{B}^{top}(n, \lambda)$ and $\mathbb{B}^{top}(n-1, \lambda+1) \cup \mathbb{B}^{top}(n-1, \lambda-1)$, as we explain shortly. The Pascal triangle formula $b_{n,\lambda} = b_{n-1,\lambda+1} + b_{n-1,\lambda-1}$ is a consequence of this bijection.
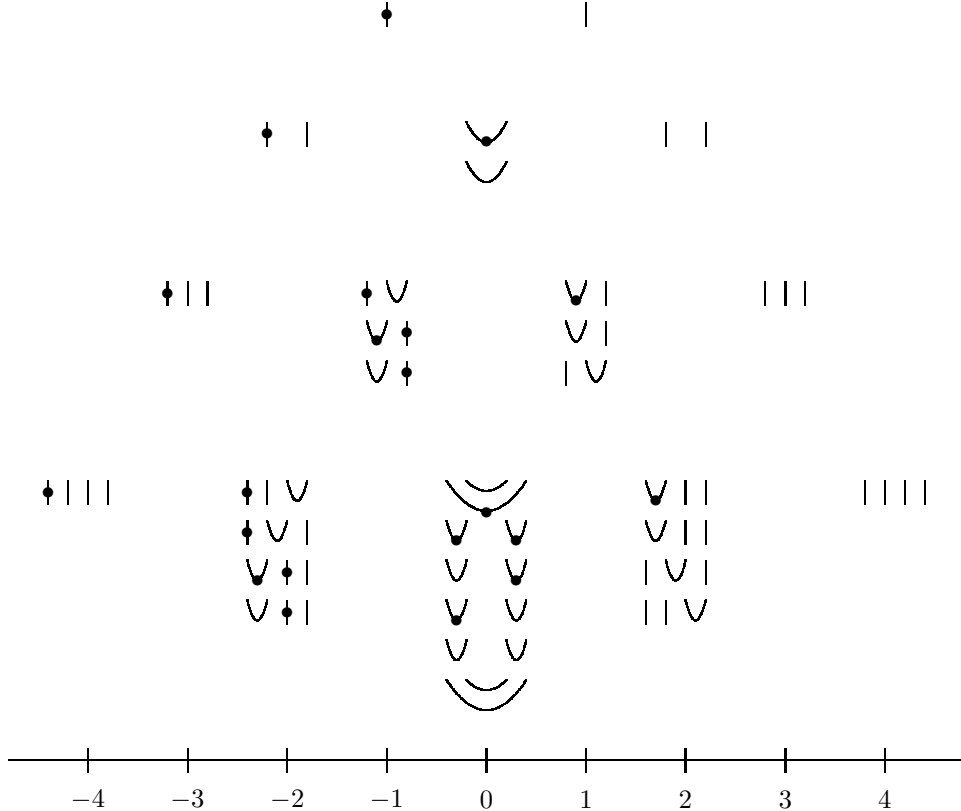

Figure 5: Bratteli diagram

For $\lambda \in \Lambda_n \setminus \{0\}$ define $\lambda^+ \in \Lambda_{n+1}$ by $\lambda^+ := \lambda \pm 1$ where the sign is positive iff $\lambda > 0$. Similarly, for $\lambda \in \Lambda_n \setminus \{0\}$ define $\lambda^- := \lambda \pm 1$ where the sign is positive iff $\lambda < 0$. Finally, if $\lambda = 0 \in \Lambda_n$ define $\lambda^+ := 1$ and $\lambda^- := -1$. With these definitions

we have for any $\lambda \in \Lambda_n$ that $\lambda^- \succ \lambda^+$ in Λ_{n+1}. In other words, the map $\lambda \mapsto \lambda^-$ moves λ closer to the central axis of the Bratteli diagram consisting of the points $\{(0, k), k = 0, 1 \ldots\}$, whereas $\lambda \mapsto \lambda^+$ takes λ away from the central axis.

The above mentioned bijection is now induced by injective maps

$$f_{n,\lambda}^+ : \mathbb{B}^{top}(n-1, \lambda) \to \mathbb{B}^{top}(n, \lambda^+), \ f_{n,\lambda}^- : \mathbb{B}^{top}(n-1, \lambda) \to \mathbb{B}^{top}(n, \lambda^-) \qquad (4.1)$$

that can be described concretely as follows. If $m \in \mathbb{B}^{top}(n-1, \lambda)$ then $f_{n,\lambda}^+$ adds an undecorated vertical line on the right hand side of m. If $\lambda \neq 0$ then $f_{n,\lambda}^-$ joins the rightmost vertical line of m with the new n'st point of the (top) edge whereas $f_{n,0}^+$ adds a decorated vertical line on the right hand side of m. Finally, by convention $f_{1,0}^+$ (resp. $f_{1,0}^-$) maps the empty diagram to the unique diagram of $\mathbb{B}^{top}(1, 1)$ (resp. $\mathbb{B}^{top}(1, -1)$).

For us the main point of this construction is that any element of $m \in \mathbb{B}^{top}(n)$ can be written uniquely as

$$m = f_{n,\lambda_n}^{\sigma_n} \ldots f_{1,0}^{\sigma_1} \emptyset \text{ where } \sigma_k \in \{+, -\} \text{ for } k = 1, \ldots, n. \qquad (4.2)$$

In other words, the sequence of signs $\{\sigma_k\}_{k=1,\ldots,n}$ uniquely determines m and hence $\mathbb{B}^{top}(n)$ is in bijection with walks on the Bratteli diagram, starting with the empty partition in position $(0, 0)$ and at the k'th step, where the walk is situated in (k, λ_k), going inwards or outwards according to the value of σ_k. We denote by $W(m)$ the walk associated with $m \in \mathbb{B}^{top}(n)$.

Let us now return to the order $\succeq$ on $\mathrm{Std}(\boldsymbol{\lambda})$ introduced above. Suppose that $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. Then $\mathfrak{s}$ also gives rise to a walk, denoted $w(\mathfrak{s})$, on the points of the Bratteli diagram. It starts in $(0, 0)$ and for $k = 0, 1, \ldots, n-1$ goes from (k, j) to $(k+1, j-1)$ if $k+1$ is located in the second component of $\mathfrak{s}$ and to $(k+1, j+1)$ if $k+1$ is located in the first component of $\mathfrak{s}$. In other words, at the k'th step the walk $w(\mathfrak{s})$ is situated in $(k, \mathfrak{s}(k))$ where $\{\mathfrak{s}(k) \,|\, k = 0, 1, \ldots, n\}$ is the sequence of integers associated with $\mathfrak{s}$ as in Definition 4.4. With this walk realization of the bitableaux, we can visualize the order $\succeq$. Indeed, let $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then $\mathfrak{s} \succeq \mathfrak{t}$ iff at each step of the two walks $w(\mathfrak{s})$ is either strictly closer than $w(\mathfrak{t})$ to the central vertical axis of the Bratteli diagram or they are at the same distance from the central axis and $w(\mathfrak{s})$ is located (weakly) to the left of $w(\mathfrak{t})$.

Let us now explain the relationship between the two walks. We denote by s the bijection $\mathbb{B}^{top}(n) \to \mathrm{Std}(\boldsymbol{n})$, $m \mapsto \mathfrak{t}_{top}(m)$, mentioned above.

Lemma 4.9. *Let $m \in \mathbb{B}^{top}(n)$. Then we have $W(m) = w(s(m))$.*

Proof: This is a consequence of Remark 4.7 and the definitions. $\qquad\square$

There is a natural surjective map $\pi : \mathbb{B}(n) \to \mathbb{T}(n)$, which sends a blob diagram m to the (n, n)-bridge obtained by deleting all decorations in m. On the other hand, $\mathbb{T}(n)$ is in bijection with pairs of two-column standard tableaux of the same shape and $\mathbb{B}(n)$ is in bijection with pairs of one-line standard bitableaux of the same shape by Definition 4.6, and so our next goal is to describe the above map π in terms of one-line bitableaux and two-column tableaux. For this we make a couple of definitions.

Definition 4.10. Suppose that $\boldsymbol{\lambda} = ((a), (b)) \in \mathrm{Bip}_1(n)$ and let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Set $\mu_1 = \max\{a, b\}$ and $\mu_2 = \min\{a, b\}$. Let μ be the two-column partition of n given by $\mu = (\mu_1, \mu_2)'$. Then we define $\tau_\mathfrak{t}$ as the unique μ-standard tableau that satisfies

k is in the second column of $\tau_{\mathfrak{t}}$ if and only if $|\mathfrak{t}(k)| < |\mathfrak{t}(k-1)|$.

We claim that $\tau_{\mathfrak{t}}$ defined in this way is a standard tableau. For this we use that a node k of the blob diagram given by $m_{\mathfrak{s}\mathfrak{t}}$ is a right endpoint in the top (resp. bottom) edge if and only if $|\mathfrak{s}(k)| < |\mathfrak{s}(k-1)|$ (resp. $|\mathfrak{t}(k)| < |\mathfrak{t}(k-1)|$), as can easily be seen by analysing Definition 4.6. In other words, $\tau_{\mathfrak{s}}$ and $\tau_{\mathfrak{t}}$ can be described as the unique two-column tableaux that satisfy $\pi(m_{\mathfrak{s}\mathfrak{t}}) = \beta_{\tau_{\mathfrak{s}}\tau_{\mathfrak{t}}}$, where π is the map defined above, and our claim follows.

For $\mathfrak{s} \in \text{Std}(\boldsymbol{\lambda})$ we let $|w(\mathfrak{s})|$ denote the walk on the Bratteli diagram that at the k'th step is located in the point $(k, |\mathfrak{s}(k)|)$. The two components of its associated bitableau are then the conjugates of the columns of $\tau_{\mathfrak{s}}$, as follows from the above.

Definition 4.11. For $\boldsymbol{\lambda} \in \text{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\boldsymbol{\lambda})$, we write $\mathfrak{s} \sim \mathfrak{t}$ if $\tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$. Thus $\mathfrak{s} \sim \mathfrak{t}$ if and only if $|\mathfrak{s}(k)| = |\mathfrak{t}(k)|$ for all $1 \leq k \leq n$.

We give a couple of Lemmas related to these definitions.

Lemma 4.12. *Suppose that $\boldsymbol{\lambda} \in \text{Bip}_1(n)$ and let $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\boldsymbol{\lambda})$. Then, $\tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$ if and only if $|\mathfrak{s}(k)| \leq |\mathfrak{t}(k)|$ for all $1 \leq k \leq n$. In particular, if $\mathfrak{s} \succeq \mathfrak{t}$ then $\tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$.*

Proof: Notice that

$$\text{Shape}(\tau_{\mathfrak{s}}|_k) = \left(\frac{k + |\mathfrak{s}(k)|}{2}, \frac{k - |\mathfrak{s}(k)|}{2} \right)'$$

$$\text{Shape}(\tau_{\mathfrak{t}}|_k) = \left(\frac{k + |\mathfrak{t}(k)|}{2}, \frac{k - |\mathfrak{t}(k)|}{2} \right)'$$

for all $1 \leq k \leq n$. Using the property of the usual dominance order that $\mu \trianglerighteq \nu \iff \nu' \trianglerighteq \mu'$ we deduce that $\tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$ if and only if $|\mathfrak{s}(k)| \leq |\mathfrak{t}(k)|$ for all $1 \leq k \leq n$, which is the first claim of the Lemma. The second claim follows now from Lemma 4.5. $\square$

Using the natural embedding $\iota : \mathbb{T}(n) \to \mathbb{B}(n)$ we obtain a walk description of the elements of $\mathbb{T}(n)$ as well. Under this description, $\mathbb{T}(n)$ corresponds to the walks on the Bratteli diagram for $b_n(m)$ that always stay in the positive half of the Bratteli diagram, including the central vertical axis.

The left action of $\mathfrak{S}_n$ on tableaux generalizes to an action of $\mathfrak{S}_n$ on bitableaux. Using it we have the following Lemma.

Lemma 4.13. *Suppose that $\boldsymbol{\lambda} \in \text{Bip}_1(n)$ and let $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\boldsymbol{\lambda})$. Suppose moreover that $\mathfrak{s} \succ \mathfrak{t}$, that $s_k \mathfrak{s} = \mathfrak{t}$ for some $s_k \in \mathfrak{S}_n$ and that $\mathfrak{s} \approx \mathfrak{t}$. Then $s_k \tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$ and $\tau_{\mathfrak{s}} \rhd \tau_{\mathfrak{t}}$.*

Proof: Note first that by the assumptions we have $\mathfrak{s}(j) = \mathfrak{t}(j)$ for all $j \neq k$. Let us first assume that $\mathfrak{s}(k+1) \geq 1$. Then $\mathfrak{s}(k) \geq 0$ since $\mathfrak{s}(j)$ changes by ± 1 when j is increased by 1. But similarly $\mathfrak{t}(k) \geq 0$ and then we must have $\mathfrak{t}(k) = \mathfrak{s}(k) + 2$ since $\mathfrak{s} \succ \mathfrak{t}$. Since k and $k+1$ are located in different components in $\mathfrak{s}$ and in $\mathfrak{t}$, this gives us the equalities

$$\mathfrak{s}(k-1) = \mathfrak{t}(k-1) = \mathfrak{s}(k) + 1 = \mathfrak{t}(k) - 1 = \mathfrak{s}(k+1) = \mathfrak{t}(k+1)$$

from which we get by Definition 4.10 that k (resp. $k+1$) is located in second (resp. first) column of $\tau_{\mathfrak{s}}$ whereas k (resp. $k+1$) is placed in first (resp. second) column of $\tau_{\mathfrak{t}}$. Since j is located in the same column of $\tau_{\mathfrak{s}}$ and $\tau_{\mathfrak{t}}$ for $j \neq k, k+1$ we now conclude that $s_k \tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$ and $\tau_{\mathfrak{s}} \rhd \tau_{\mathfrak{t}}$, as needed.

The case $\mathfrak{s}(k+1) \leq -1$ is treated similarly and so the only remaining case is $\mathfrak{s}(k+1) = 0$. Then $\mathfrak{t}(k+1) = \mathfrak{s}(k-1) = \mathfrak{t}(k-1) = 0$. Moreover since $\mathfrak{s} \succ \mathfrak{t}$ we have $\mathfrak{s}(k) = -1$ and $\mathfrak{t}(k) = 1$. But this implies that $\mathfrak{s} \sim \mathfrak{t}$, finishing the proof. $\qquad\square$

Definition 4.14. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we say that "$\mathfrak{s}$ has a hook at position k" if $\mathfrak{s}(k-1) = \mathfrak{s}(k+1) = \mathfrak{s}(k) \pm 1$ where $1 \leq k \leq n-1$. Moreover we say that "$\mathfrak{t}$ is obtained from $\mathfrak{s}$ by making a hook at position k smaller" if $\mathfrak{s}(j) = \mathfrak{t}(j)$ for $j \neq k$, $\mathfrak{s}(k) = \mathfrak{t}(k) \pm 2$ and $\mathfrak{s} \succ \mathfrak{t}$.

The last condition can also be written as $s_k \mathfrak{s} = \mathfrak{t}$ and $\mathfrak{s} \succ \mathfrak{t}$. Geometrically, if $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by making a hook at position k smaller then $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by either replacing a configuration of three consecutive points in $w(\mathfrak{t})$ forming a "$\langle$" by a configuration "$\rangle$" at these three points, or reversely, depending on which side of the Bratteli diagram the configuration is located.

Lemma 4.15. *For* $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ *we define* $\mathfrak{d}(\mathfrak{t})$ *as the element of* $\mathfrak{S}_n$ *that satisfies* $\mathfrak{t} = \mathfrak{d}(\mathfrak{t})\mathfrak{t}^{\boldsymbol{\lambda}}$. *Then* $\mathfrak{d}(\mathfrak{t})$ *can be written as product of simple transpositions* $\mathfrak{d}(\mathfrak{t}) = s_{i_k}s_{i_{k-1}}\ldots s_{i_1}$ *such that* $s_{i_j}\ldots s_{i_1}\mathfrak{t}^{\boldsymbol{\lambda}}$ *is standard and such that* $s_{i_j}s_{i_{j-1}}\ldots s_{i_1}\mathfrak{t}^{\boldsymbol{\lambda}} \prec s_{i_{j-1}}\ldots s_{i_1}\mathfrak{t}^{\boldsymbol{\lambda}}$ *for all* $1 \leq j \leq k$.

Proof: This can be seen via the walk realization of $\mathrm{Std}(\boldsymbol{\lambda})$. Indeed the walk $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ first zigzags on and off the central vertical line of the Bratteli diagram, using the sign $-$ an even number of times, and then finishes using the sign $+$ repeatedly, if $\boldsymbol{\lambda}$ is located in the positive half, or using once the sign $-$ followed by the sign $+$ repeatedly, if $\boldsymbol{\lambda}$ is located in the negative half. Figure 6 shows an example of such walks.

Figure 6: Walk $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ for $\boldsymbol{\lambda} = ((2), (6))$ (left) and $\boldsymbol{\lambda} = ((6), (2))$ (right).

This walk can be converted into $w(\mathfrak{t})$ through a series of k transformations, say, where at each step the new walk is obtained from the previous one by making a hook at position j smaller, for some j. At tableau level, each of these transformations is given by the action of a simple transposition s_j. The Lemma follows from this. $\square$

Remark 4.16. Although we do not need it directly, we note that $l(\mathfrak{d}(\mathfrak{t})) = k$ and that the expression $\mathfrak{d}(\mathfrak{t}) = s_{i_k}s_{i_{k-1}}\ldots s_{i_1}$ is reduced.

5. Jucys-Murphy elements for b_n.

In Corollary 3.6, we gave a new (graded) presentation for $b_n(m)$, while in the previous section we described the diagram basis for the blob algebra. Unfortunately, it seems nontrivial to express the homogeneous generators in terms of the diagram basis of $b_n(m)$. However, it turns out that a graded cellular basis for $b_n(m)$ can be constructed from a precise description of the KLR idempotents in $b_n(m)$. Inspired by the work of Hu and Mathas [11], we shall obtain in the next section an expression for them building on the results from [17]. A key point for this is to make

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$b_n(m)$ fit into the general setting of an algebra with Jucys-Murphy elements. This is the main goal of this section.

5.1. Cellular algebras.

Before defining the concept of an algebra with Jucys-Murphy elements, we first recall the definition of a cellular algebra, which was first given by Graham and Lehrer in [8] in order to provide a common framework for a series of algebras that appear in non-semisimple representation theory.

Definition 5.1. Let R be an integral domain. Suppose that A is an R-algebra which is free of finite rank over R. Suppose that $(\Lambda, \geq)$ is a poset and that for each $\lambda \in \Lambda$ there is a finite set $T(\lambda)$ and elements $c_{\mathfrak{s}\mathfrak{t}}^\lambda \in A$ such that

$$\mathcal{C} = \{c_{\mathfrak{s}\mathfrak{t}}^\lambda \mid \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$$

is a basis of A. The pair $(\mathcal{C}, \Lambda)$ is a cellular basis of A if

(i) The R-linear map $* : A \to A$ determined by $(c_{\mathfrak{s}\mathfrak{t}}^\lambda)^* = c_{\mathfrak{s}\mathfrak{t}}^\lambda$, for all $\lambda \in \Lambda$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, is an algebra anti-automorphism of A

(ii) If $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for some $\lambda \in \Lambda$, and $a \in A$ then there exist scalars $r_{\mathfrak{u}} \in R$ such that

$$ac_{\mathfrak{s}\mathfrak{t}}^\lambda \equiv \sum_{\mathfrak{u} \in T(\lambda)} r_{\mathfrak{u}} c_{\mathfrak{u}\mathfrak{t}}^\lambda \mod A^\lambda$$

where A^λ is the R-submodule of A spanned by $\{c_{\mathfrak{a}\mathfrak{b}}^\mu \mid \mu > \lambda; \mathfrak{a}, \mathfrak{b} \in T(\mu)\}$

If A has a cellular basis we say that A is a cellular algebra.

Note that $r_{\mathfrak{u}}$ depends on $\mathfrak{v}$, $\mathfrak{s}$ and a, what is important is that $r_{\mathfrak{u}}$ does not depend on $\mathfrak{t}$. Now suppose that A is a $\mathbb{Z}$-graded R-algebra and that each $c_{\mathfrak{s}\mathfrak{t}}^\lambda$ is homogeneous. If there exists a function

$$\deg : \coprod_{\lambda \in \Lambda} T(\lambda) \to \mathbb{Z}$$

such that $\deg c_{\mathfrak{s}\mathfrak{t}}^\lambda = \deg \mathfrak{s} + \deg \mathfrak{t}$, for $\lambda \in \Lambda$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, we say that $\mathcal{C} = \{c_{\mathfrak{s}\mathfrak{t}}^\lambda \mid \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$ is a graded cellular basis of A. If A has a graded cellular basis we say that A is a graded cellular algebra.

Definition 5.2. Suppose that A is a cellular algebra with cellular basis $(\mathcal{C}, \Lambda)$ as in the Definition 5.1 and fix $\lambda \in \Lambda$. Then the cell module C^λ is the left A-module which is free as an R-module with basis $\{c_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$ and where the action A on C^λ is given by

$$ac_{\mathfrak{t}}^\lambda = \sum_{\mathfrak{u} \in T(\lambda)} r_{\mathfrak{u}} c_{\mathfrak{u}}^\lambda$$

where $r_{\mathfrak{u}}$ is the element of R that appears in the Definition 5.1 (ii).

If in the above definition we assume that A is a graded cellular algebra, then the cell modules are graded modules. In this case we have the direct sum decomposition

$$C^\lambda = \bigoplus_{z \in \mathbb{Z}} C_z^\lambda$$

where C_z^λ is the free R-module with basis $\{c_{\mathfrak{t}}^\lambda \mid \mathfrak{t} \in T(\lambda)$ and $\deg \mathfrak{t} = z\}$.

For $\lambda \in \Lambda$ there is a symmetric and associative bilinear form $\langle \cdot, \cdot \rangle_\lambda$ on C^λ determined by

$$c^\lambda_{\mathfrak{a}\mathfrak{s}} c^\lambda_{\mathfrak{t}\mathfrak{b}} \equiv \langle c^\lambda_\mathfrak{s}, c^\lambda_\mathfrak{t} \rangle_\lambda c^\lambda_{\mathfrak{a}\mathfrak{b}} \mod A^\lambda$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{s}, \mathfrak{t} \in T(\lambda)$. Define the radical of C^λ by

$$\operatorname{rad} C^\lambda = \{x \in C^\lambda \mid \langle x, y \rangle_\lambda = 0 \text{ for all } y \in C^\lambda\}$$

It is easy to see that rad C^λ is an A-submodule of C^λ (See [16, Proposition 2.9]), and that also rad C^λ is a graded submodule of C^λ (See [11, Lemma 2.7]) . Define $D^\lambda = C^\lambda/\operatorname{rad} C^\lambda$. By definition D^λ is a graded A-module.

In general, for a cellular algebra A we define $\Lambda_0 = \{\lambda \in \Lambda \mid D^\lambda \neq 0\}$. The following theorem classifies the simple A-modules over a field.

Theorem 5.3. *Suppose that R is a field. Then $\{D^\lambda \mid \lambda \in \Lambda_0\}$ is a complete set of pairwise non-isomorphic simple A-modules.*

It is well known that the Temperley-Lieb algebra ([8, Example 1.4]) and the blob algebra ([9, page 7]) are cellular, in fact the diagram bases introduced in the above section are cellular in both cases. We now recall the various elements of these cellular structures.

For the Temperley-Lieb algebra, according to the notation introduced in Definition 5.1, we take $\Lambda = \operatorname{Par}_2(n)$, ordered by dominance. We take $T(\lambda) = \operatorname{Std}(\lambda)$, for all $\lambda \in \Lambda$ and for $\sigma, \tau \in T(\lambda)$ we define $c^\lambda_{\sigma\tau} = \beta_{\sigma\tau}$.

Similarly, for the blob algebra we take $\Lambda = \operatorname{Bip}_1(n)$, ordered by $\succeq$ introduced in the previous section. Set $T(\boldsymbol{\lambda}) = \operatorname{Std}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \operatorname{Bip}_1(n)$. Given $\mathfrak{s}, \mathfrak{t} \in T(\boldsymbol{\lambda})$ define $c^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{s}\mathfrak{t}}$.

For $\boldsymbol{\lambda} \in \Lambda$, let $b^{\boldsymbol{\lambda}}_n(m)$ be the ideal of $b_n(m)$ spanned by the set

$$\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu}) \; ; \; \boldsymbol{\mu} \succ \boldsymbol{\lambda}\}.$$

In the cases of $b^\mathcal{O}_n(m)$ and $b^K_n(m)$ we write $b^{\mathcal{O},\boldsymbol{\lambda}}_n(m)$ and $b^{K,\boldsymbol{\lambda}}_n(m)$ for the ideals.

Since we are assuming that $q + q^{-1} \neq 0$ we get that the bilinear forms $\langle \cdot, \cdot \rangle_\lambda$ are all nonzero, in the Temperley-Lieb case as well as the blob algebra case. From this we get from remark (3.10) of [8] that both algebras are quasi-hereditary and that the cell modules are standard modules in the sense of quasi-hereditary algebras.

5.2. Jucys-Murphy elements.

We are now in position to give the definition of an algebra with Jucys-Murphy elements. It provides an abstract setting for carrying out much of Murphy's theory for Young's seminormal form. Assume that A is a cellular algebra with cellular basis

$$\mathcal{C} = \{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$$

as in Definition 5.1. Assume furthermore that each $T(\lambda)$ is a poset with respect to an order $<_\lambda$, or just $<$ for simplicity. The following definition is taken from [17].

Definition 5.4. A family of Jucys-Murphy (JM) elements for A is a set $\{L_1, \ldots, L_k\}$ of commuting elements of A together with a set of scalars,

$$\{c_\mathfrak{s}(i) \in R \mid \mathfrak{s} \in T(\lambda), \lambda \in \Lambda \text{ and } 1 \leq i \leq k\}$$

such that for $i = 1, \dots, k$ we have $L_i^* = L_i$ and, for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \Lambda$,

$$L_i c_{\mathfrak{s}\mathfrak{t}}^\lambda \equiv c_s(i) c_{\mathfrak{s}\mathfrak{t}}^\lambda + \sum_{\mathfrak{v} > \mathfrak{s}} r_{\mathfrak{s}\mathfrak{v}} c_{\mathfrak{v}\mathfrak{t}}^\lambda \mod A^\lambda$$

for some $r_{\mathfrak{s}\mathfrak{v}} \in R$ (which depends on i). We call $c_{\mathfrak{s}}(i)$ the content of $\mathfrak{s}$ at i.

The purpose of this section is now to apply this definition to $b_n(m)$. By Theorem 2.6 we have a homomorphism from $\mathcal{H}_n(m)$ onto $b_n(m)$, it maps the elements $L_k \in \mathcal{H}_n(m)$ to

$$(U_{k-1} + q) \dots (U_1 + q)((q - q^{-1})U_0 + q^m)(U_1 + q) \dots (U_{k-1} + q) \in b_n(m).$$

We shall use the same notation L_k for this element of $b_n(m)$. It satisfies the following commutation rules with the U_i

$$L_k U_i = U_i L_k \qquad\qquad\qquad \text{if } k \neq i, i+1 \qquad (5.1)$$
$$(U_k + q^{-1})L_{k+1} = L_k(U_k + q) \qquad\qquad \text{for } 1 \leq k < n. \qquad (5.2)$$
$$L_{k+1}(U_k + q^{-1}) = (U_k + q)L_k \qquad\qquad \text{for } 1 \leq k < n. \qquad (5.3)$$

It is known that the L_k are a family of JM-elements for $\mathcal{H}_n(m)$ with respect to the cellular basis used for example in [11], in which $<_\lambda$ is the dominance order on bitableaux. One might now hope that the set $\{L_1, \dots L_n\}$ is also a family of JM-elements for $b_n(m)$. That this should be the case is not at all obvious. Indeed, the concept of a family of JM-elements depends heavily on the underlying cellular basis and a cellular algebra may in general be endowed with several, completely different, cellular bases with different orders. For example the conjectures of Bonnafé, Geck, Iancu and Lam in [1], indicate that Lusztig's theory of cells for unequal parameters should give rise to a cellular basis on $\mathcal{H}_n(m)$ for each choice of a weight function on the Coxeter group of type B, in dependence of a parameter r. In this setting only the asymptotic case $r > n$ corresponds to the dominance order on $T(\lambda)$. On the contrary, in [21] it is shown that the cell structure on $b_n(m)$ corresponds to the other extreme case $r = 0$ under restriction to $\mathrm{Bip}_1(n)$.

In this section we shall show that in fact $\{L_1, \dots L_n\}$ do form a family of JM-elements for $b_n(m)$ where the poset structure $T(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ is the one defined above. Even more, using the surjection $\mathcal{H}_n^{\mathcal{O}}(m) \longrightarrow b_n^{\mathcal{O}}(m)$ given in Theorem 2.7, we define elements $\{L_1, \dots, L_n\}$ of $b_n^{\mathcal{O}}(m)$ using the same formula as before and we show that these form a family of JM-elements for $b_n^{\mathcal{O}}(m)$ with respect to $\{m_{\mathfrak{s}\mathfrak{t}}\}$, considered as elements of $b_n^{\mathcal{O}}(m)$.

Definition 5.5. *Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Let k be an integer with $1 \leq k \leq n$. Define the content of $\mathfrak{t}$ at k to be the scalar given by*

$$c_{\mathfrak{t}}(k) = \begin{cases} q^{2(c-1)}Q & \text{if} \quad l = 1 \\ q^{2(c-1)}Q^{-1} & \text{if} \quad l = 2 \end{cases}$$

where $(1, c, l)$ is the unique node in $[\boldsymbol{\lambda}]$ such that $\mathfrak{t}(1, c, l) = k$. In other words, $c_{\mathfrak{t}}(k)$ is an element of either $\mathcal{O}, \mathbb{C}(q, Q)$ or $\mathbb{C}$, depending on the context. In the $\mathbb{C}$ case, we shall also write $r_{\mathfrak{t}}(k) := c_{\mathfrak{t}}(k)$ and refer to it as the residue of k.

Lemma 5.6. *Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and let k be an integer with $1 \leq k \leq n$. Then we have the identity*

$$L_k m_{\mathfrak{t}^{\boldsymbol{\lambda}} \mathfrak{t}^{\boldsymbol{\lambda}}} \equiv c_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k) m_{\mathfrak{t}^{\boldsymbol{\lambda}} \mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

Similar statements hold over $b_n^{\mathcal{O}}(m)$ and $b_n^{\mathcal{K}}(m)$.

Proof: Using the description of $t^{\boldsymbol{\lambda}}$ given after Definition 4.3 together with Definition 4.6 we find that the diagram corresponding to $m_{t^{\boldsymbol{\lambda}} t^{\boldsymbol{\lambda}}}$ is one of the diagrams that appear in Figure 7. But then the statement of the Lemma is equation (25) of [6, Lemma 7.1]. (Note that there is an error in equation (25) as presented in *loc. cit.* As a matter of fact, to get the correct expressions one should subtract 2 from all appearing exponents of x since the relation between L_i and L'_i introduced two pages earlier should be corrected the same way). $\qquad\square$
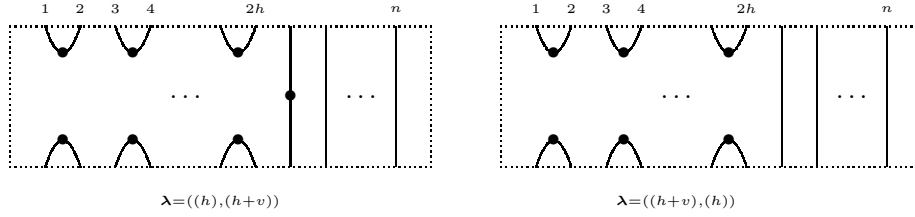
Figure 7: $m_{t^{\boldsymbol{\lambda}} t^{\boldsymbol{\lambda}}}$

Our proof that the $\{L_k\}$ form a family of JM-elements shall be a downwards induction over the partial order $\succeq$ with the preceding Lemma providing the induction basis. To obtain the inductive step we need to understand the relationship between the action of U_i and $\succeq$ and hence we would like to have a formula for the action of U_k in terms of walks on the Bratteli diagram. In general there is no such simple formula. On the other hand, there is one situation where the action of U_k is particularly easy to visualize.

Lemma 5.7. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Assume moreover that $s_k\mathfrak{s} = \mathfrak{t}$ for the simple transposition s_k and that $\mathfrak{s} \succ \mathfrak{t}$ or equivalently, that $w(\mathfrak{t})$ is obtained from $w(\mathfrak{s})$ by making a hook at position k smaller. Then the following relation holds in $b_n(m)$*

$$U_k m_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} = \left\{ \begin{array}{ll} m_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} & if \quad \mathfrak{s} \not\sim \mathfrak{t} \\ y_e m_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} & if \quad \mathfrak{s} \sim \mathfrak{t}. \end{array} \right.$$

Similar formulas hold over $\mathcal{O}, K$ and in the Temperley-Lieb algebra (corresponding to $\mathfrak{s} \not\sim \mathfrak{t}$).

Proof: This is an immediate consequence of the definition of the maps $f_{n,k}^{\sigma}$. $\quad\square$

The next three Lemmas are preparations for Lemma 5.11.

Lemma 5.8. *Suppose that $\lambda \in Par_2(n)$ and $\sigma, \tau, u \in \mathrm{Std}(\lambda)$. Suppose moreover that $u \rhd \sigma \rhd \tau$ and that $s_k\sigma = \tau$ for some k. Let $v \in \mathrm{Std}(\lambda)$ be chosen such that $U_k\beta_{ut^{\lambda}} = r\beta_{vt^{\lambda}} \mod Tl^{\lambda}$ for some scalar $r \in \mathbb{C}$ (such v always exists by the diagrammatical realization of the Temperley-Lieb algebra and its cell modules). Then, if r is nonzero we have that $v \rhd \tau$.*

Proof: We identify σ, τ and u with their walks $w(\sigma), w(\tau)$ and $w(u)$ on the Bratteli diagram for Tl_n, and also with their corresponding sign sequences. Then the sign sequences for σ and τ are the same except at the k'th and $k+1$'st positions where the sequence for σ has $-, +$ whereas the sequence for τ has $+, -$. On the other hand, for u all four possibilities of signs may occur at these positions, apriori, and so we proceed by a case by case analysis.

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The first case to analyse is the case where the signs for u are $+, -$ at these positions. In this case we get $v = u$ (and $r := -[2]$), and so the claim of the Lemma follows from the assumptions. The next case is the one where the signs are $+, +$ at positions k and $k + 1$. On the diagrammatic level we have three options for the top edge of $\beta_{u\tau^\lambda}$, illustrated in Figure 8.
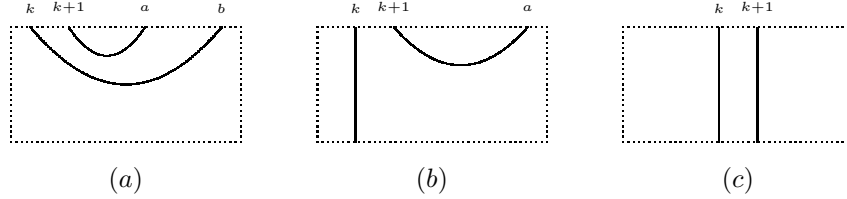

$$(a) \qquad\qquad (b) \qquad\qquad (c)$$

Figure 8: Top edge of $\beta_{u\tau^\lambda}$

In the subcase (a), the signs for u at positions k, $k + 1$, a and b are $+, +, -$ and $-$, respectively. For v the signs in these positions are $+, -, +$ and $-$, whereas the signs for v and u agree at all other positions. The claim follows from this. The subcase (b) is treated similarly. Finally, in the subcase (c) we have $r = 0$, contrary to the assumptions.

The third case is the one where the signs for u are $-, +$ at the positions $k, k+1$. In that case, at the diagrammatic level, k is connected to a point $a < k$ whereas $k + 1$ is either connected to $b > k + 1$ or it is the upper endpoint of a vertical line. In both cases, we find that the sign sequence for v is the same as the one for u, except at positions $k, k + 1$ where it becomes $+, -$. But by the assumptions, u differs from σ in at least one position and the result follows in this case as well. Note that this is the only case in which $u \rhd v$.

The last case is the one where the signs for u at the positions $k, k+1$ are $-, -$. In this case, k is connected to a and $k + 1$ to b and $b < a < k < k + 1$. Moreover the signs for u at these positions are $+, +, -, -$. But then the signs for v at these positions are $+, -, +, -$ whereas the signs for v and u agree at all other positions. The claim follows from this. $\qquad\qquad\square$

Lemma 5.9. *Suppose that $\mu \in Par_2(n)$. Let $\sigma, \tau \in Std(\mu)$. Assume that $U_k\beta_{\sigma\tau^\mu} \equiv \alpha\beta_{\tau\tau^\mu} \mod TL_n^\mu$, with $\alpha \neq 0$ and $1 \leq k < n$. Then, $\tau \unrhd \sigma$, or $\sigma \rhd \tau$ and $s_k\sigma = \tau$.*

Proof: The result follows by a case by case analysis, similar to that given in the proof of the previous Lemma. $\qquad\qquad\square$

Lemma 5.10. *Let $\boldsymbol\lambda \in Bip_1(n)$ and $\mathfrak{u} \in Std(\boldsymbol\lambda)$. Assume that $U_k m_{\mathfrak{u}\mathfrak{t}^\lambda} \equiv \alpha m_{\mathfrak{v}\mathfrak{t}^\lambda} \mod b_n^{\boldsymbol\lambda}(m)$, for $\alpha \in \mathbb{C}$ and $\mathfrak{v} \in Std(\boldsymbol\lambda)$ (such $\mathfrak{v}$ always exists by the diagrammatic realization of $b_n(m)$). Suppose moreover that α is nonzero and that the node j is covered in the top edge of $m_{\mathfrak{u}\mathfrak{t}^\lambda}$, but uncovered in the top edge of $m_{\mathfrak{v}\mathfrak{t}^\lambda}$. Then $|\mathfrak{u}(j)| = |\mathfrak{v}(j)| \neq 0$ if and only if $j = k$. Similar statements hold over $\mathcal{O}$ and K.*

Proof: In order for the action of U_k to transform a covered node j in the top edge of $m_{\mathfrak{u}\mathfrak{t}^\lambda}$ to an uncovered node in the top edge of $m_{\mathfrak{v}\mathfrak{t}^\lambda}$, the diagram of $m_{\mathfrak{u}\mathfrak{t}^\lambda}$ must be one of those shown in the below Figure 9 with the position of j shown in each case. Using this classification, the Lemma follows from Definition 4.6.

$$\square$$

We can now finally prove the property of the order $\succ$ that makes our induction work. It is a generalization to the blob algebra case of Lemma 5.8, and in fact we shall deduce it from that Lemma.

$$k \leq j < a \qquad k \leq j \leq k+1 \qquad a < j \leq k+1 \qquad k \leq j < a$$
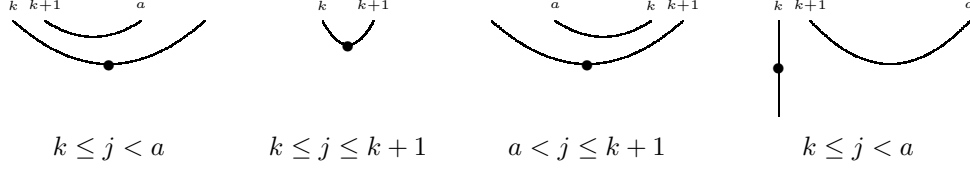
Figure 9: Possibilities for $m_{\mathfrak{u}\mathfrak{t}^\lambda}$

Lemma 5.11. *Suppose that* $\boldsymbol{\lambda} = ((a),(b)) \in Bip_1(n)$ *and* $\mathfrak{s},\mathfrak{t},\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda})$. *Suppose furthermore that* $s_k\mathfrak{s} = \mathfrak{t}$ *and that* $\mathfrak{u} \succ \mathfrak{s} \succ \mathfrak{t}$. *Let* $\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ *be chosen such that* $U_k m_{\mathfrak{u}\mathfrak{t}^\lambda} = r m_{\mathfrak{v}\mathfrak{t}^\lambda} \mod b_n^\lambda(m)$ *for some scalar* $r \in \mathbb{C}$. *Then, if* r *is nonzero we have that* $\mathfrak{v} \succ \mathfrak{t}$. *Similar statements are valid for* $b_n^{\mathcal{O}}(m)$ *and* $b_n^K(m)$.

Proof: Set $\mu_1 = \max\{a,b\}$, $\mu_2 = \min\{a,b\}$ and let $\mu = (\mu_1,\mu_2)'$. Then $\mu \in \mathrm{Par}_2(n)$ and in the Temperley-Lieb algebra we have that

$$U_k \beta_{\tau_\mathfrak{u} \tau^\mu} = r_1 \beta_{\tau_\mathfrak{v} \tau^\mu} \mod Tl^\mu$$

where $\tau_\mathfrak{u}, \tau_\mathfrak{v}$ are as in Definition 4.10 and $r_1 \neq 0$; indeed $r_1 = -[2]$ if $r = -[2]$ or if $r = y_e$, and $r_1 = 1$ if $r = 1$. Moreover, by Lemma 4.12 we have that $\tau_\mathfrak{u} \unrhd \tau_\mathfrak{s} \unrhd \tau_\mathfrak{t}$.

Case 1 ($\tau_\mathfrak{u} \rhd \tau_\mathfrak{s} \rhd \tau_\mathfrak{t}$). In this case we have by Lemma 4.13 that $s_k\tau_\mathfrak{s} = \tau_\mathfrak{t}$ and then Lemma 5.8 gives that $\tau_\mathfrak{v} \rhd \tau_\mathfrak{t}$. Now by Lemma 4.5, in order to prove that $\mathfrak{v} \succ \mathfrak{t}$ it is enough to show that

$$|\mathfrak{v}(j)| = |\mathfrak{t}(j)| \text{ implies } \mathfrak{v}(j) \leq \mathfrak{t}(j). \tag{5.4}$$

Hence, assume that $|\mathfrak{v}(j)| = |\mathfrak{t}(j)|$, but $\mathfrak{t}(j) < 0$ and $\mathfrak{v}(j) > 0$ for some $1 \leq j \leq n$. We now split this case into two subcases according to Lemma 5.9, that is, $\tau_\mathfrak{v} \unrhd \tau_\mathfrak{u}$ or, $\tau_\mathfrak{u} \rhd \tau_\mathfrak{v}$ and $s_k\tau_\mathfrak{u} = \tau_\mathfrak{v}$. First, we assume that $\tau_\mathfrak{v} \unrhd \tau_\mathfrak{u}$. Then we get from $\mathfrak{u} \succ \mathfrak{s} \succ \mathfrak{t}$ and Lemma 4.5 that $\mathfrak{u}(j) = \mathfrak{s}(j) = \mathfrak{t}(j)$ and so we get $\mathfrak{u}(j) < 0$, $\mathfrak{v}(j) > 0$ and $|\mathfrak{u}(j)| = |\mathfrak{v}(j)|$. From this we conclude via Lemma 5.10 that $j = k$, hence that $\mathfrak{s}(k) = \mathfrak{t}(k)$, which is impossible because $s_k\mathfrak{s} = \mathfrak{t}$.

So we can assume that $\tau_\mathfrak{u} \rhd \tau_\mathfrak{v}$ and $s_k\tau_\mathfrak{u} = \tau_\mathfrak{v}$. Assume first that $k \neq j$. Then $\mathfrak{s}(j) = \mathfrak{t}(j)$ whereas $|\mathfrak{u}(j)| = \mathfrak{v}(j)$. But $\mathfrak{u} \succ \mathfrak{s}$ implies $\mathfrak{u}(j) = \mathfrak{s}(j)$ and so we have $\mathfrak{u}(j) = \mathfrak{s}(j) = \mathfrak{t}(j)$. As before this implies via Lemma 5.10 that $j = k$, contradiction. We then assume $j = k$. Then $w(\mathfrak{t})$ is obtained from $w(\mathfrak{s})$ by making a hook at position j bigger and so we have $\mathfrak{s}(j) = \mathfrak{t}(j)+2 \leq 0$ and $\mathfrak{s}(j\pm 1) = \mathfrak{t}(j\pm 1) = \mathfrak{t}(j)+1$. On the other hand, $\tau_\mathfrak{u} \rhd \tau_\mathfrak{v}$ and $s_k\tau_\mathfrak{u} = \tau_\mathfrak{v}$ imply that $|\mathfrak{u}(j)| = \mathfrak{v}(j) - 2$ which combined with $\mathfrak{u} \succ \mathfrak{s}$ gives $\mathfrak{u}(j) = \mathfrak{s}(j)$ and then also $\mathfrak{u}(j \pm 1) = \mathfrak{s}(j \pm 1)$, since $\mathfrak{u}(i)$ at most changes by ± 1 at each step. But then Lemma 5.7 implies that $\mathfrak{v}(j) = \mathfrak{t}(j)$, contradiction.

Case 2 ($\tau_\mathfrak{u} \unrhd \tau_\mathfrak{s} = \tau_\mathfrak{t}$). By the assumptions $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by making a hook at position k smaller. Moreover, since $\tau_\mathfrak{s} = \tau_\mathfrak{t}$ this hook is located on the central vertical axis of the Bratteli diagram, that is $\mathfrak{t}(k-1) = \mathfrak{s}(k-1) = \mathfrak{t}(k+1) = \mathfrak{s}(k+1) = 0$. But then, since $\mathfrak{u} \succ \mathfrak{s}$, we have necessarily that $\mathfrak{u}(k-1) = \mathfrak{u}(k+1) = 0$, $\mathfrak{u}(k) = -1$ which implies via Lemma 5.7 that $\mathfrak{v}$ is obtained from $\mathfrak{u}$ by making a hook at position k smaller. Hence we get $\mathfrak{v} \succ \mathfrak{t}$ as claimed.

Case 3 ($\tau_\mathfrak{u} = \tau_\mathfrak{s} \unrhd \tau_\mathfrak{t}$). By the hypothesis in this case we have $\tau_\mathfrak{v} = \tau_\mathfrak{t}$. Recall that at the Bratteli diagram level this implies that at each step the walks $w(\mathfrak{t})$ and $w(\mathfrak{v})$ are either equal or mirror images under the reflection through the central vertical axis of the Bratteli diagram. So, in order to prove the Lemma in this case we must

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prove that whenever the path $w(\mathfrak{t})$ is on the negative side of the Bratteli diagram, the path $w(\mathfrak{v})$ is also on the negative part. In terms of the sequence of integers the last condition is equivalent to

$$\mathfrak{t}(j) < 0 \text{ implies } \mathfrak{v}(j) < 0 \qquad (5.5)$$

for all $1 \leq j \leq n$. Suppose by contradiction that (5.5) is not true for some $1 \leq j \leq n$. Therefore, $\mathfrak{t}(j) < 0 < \mathfrak{v}(j)$ for some $1 \leq j \leq n$. Using the fact that $\mathfrak{s}(j) = \mathfrak{t}(j)$, for all $j \neq k$, $\tau_{\mathfrak{u}} = \tau_{\mathfrak{s}}$ and $\tau_{\mathfrak{v}} = \tau_{\mathfrak{t}}$, we can conclude via Remark 4.7 and Lemma 5.10 that $j = k$. Hence, at step k the walk $w(\mathfrak{t})$ (resp. $w(\mathfrak{v})$) is on the negative side of Bratteli diagram and (5.5) is true for all $j \neq k$. This implies that $\mathfrak{t}(k-1) = \mathfrak{t}(k+1) = 0$ and $\mathfrak{t}(k) = -1$. But this is impossible because $\mathfrak{s} \succ \mathfrak{t}$ and $s_k \mathfrak{s} = \mathfrak{t}$. This completes the proof of the Lemma. □

We are now in position to prove the triangularity property for $\{L_1, \ldots, L_n\}$. It follows from it that the set $\{L_1, \ldots, L_n\}$ is a family of JM-elements for the blob algebra with respect to the order $\succ$.

Theorem 5.12. *Suppose that* $\boldsymbol{\lambda} \in Bip_1(n)$ *and* $s, t \in \mathrm{Std}(\boldsymbol{\lambda})$. *Then*

$$L_k m_{\mathfrak{s}\mathfrak{t}} = c_{\mathfrak{s}}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{u} \succ \mathfrak{s}}} a_{\mathfrak{u}} m_{\mathfrak{u}\mathfrak{t}} \mod b_n^{\boldsymbol{\lambda}}(m)$$

for some scalars $a_{\mathfrak{u}}$. *A similar statements holds for* $b_n^{\mathcal{O}}(m)$ *and* $b_n^K(m)$.

Proof: By the cellularity of the diagram basis, the statement of the Lemma is independent of $\mathfrak{t}$. We proceed by induction on the order $\succeq$. The induction basis $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$ is provided by Lemma 5.6. Assume now that $\mathfrak{s} \neq \mathfrak{t}^{\boldsymbol{\lambda}}$. Then we can find i and $\mathfrak{s}'$ such that $\mathfrak{s}' \succ \mathfrak{s}$ and $s_i \mathfrak{s}' = \mathfrak{s}$. By the inductive hypothesis the Theorem is valid for $\mathfrak{s}'$. We first assume that $\mathfrak{s} \sim \mathfrak{s}'$ and $k \neq i, i+1$. Using Lemma 5.7 and the commutation rule (5.1) we then get

$$L_k m_{\mathfrak{s}\mathfrak{t}} = L_k U_i m_{\mathfrak{s}'\mathfrak{t}} = U_i L_k m_{\mathfrak{s}'\mathfrak{t}} = c_{\mathfrak{s}'}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{u} \succ \mathfrak{s}'}} a_{\mathfrak{u}} U_i m_{\mathfrak{u}\mathfrak{t}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

On the other hand, by the previous Lemma the sum is a linear combination of elements of the form $m_{\mathfrak{u}\mathfrak{t}}$ where $\mathfrak{u} \succ \mathfrak{s}$ and since $c_{\mathfrak{s}}(k) = c_{\mathfrak{s}'}(k)$ we are done in this case.

If $\mathfrak{s} \sim \mathfrak{s}'$ and $k \neq i, i+1$ we find similarly

$$L_k m_{\mathfrak{s}\mathfrak{t}} = y_e^{-1} L_k U_i m_{\mathfrak{s}'\mathfrak{t}} = y_e^{-1} U_i L_k m_{\mathfrak{s}'\mathfrak{t}} = c_{\mathfrak{s}'}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{u} \succ \mathfrak{s}'}} y_e^{-1} a_{\mathfrak{u}} U_i m_{\mathfrak{u}\mathfrak{t}} \mod b_n^{\boldsymbol{\lambda}}(m)$$

and may conclude the same way as before. We next treat the case $\mathfrak{s} \sim \mathfrak{s}'$ and $i = k$ where we find, using the commutation rule (5.2) that

$$L_k m_{\mathfrak{s}\mathfrak{t}} = L_k U_k m_{\mathfrak{s}'\mathfrak{t}} = L_k (U_k + q - q) m_{\mathfrak{s}'\mathfrak{t}} = (U_k + q^{-1}) L_{k+1} m_{\mathfrak{s}'\mathfrak{t}} - q L_k m_{\mathfrak{s}'\mathfrak{t}}.$$

By the inductive hypothesis, $L_k m_{\mathfrak{s}'\mathfrak{t}}$ and $L_{k+1} m_{\mathfrak{s}'\mathfrak{t}}$ are linear combination of elements of the form $m_{\mathfrak{u}\mathfrak{t}}$ where $\mathfrak{u} \succ \mathfrak{s}$ and hence we find, using the inductive hypothesis and Lemma 5.7 once more, that $L_k m_{\mathfrak{s}\mathfrak{t}}$ is equal to

$$U_k L_{k+1} m_{\mathfrak{s}'\mathfrak{t}} = c_{\mathfrak{s}'}(k+1) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{u} \succ \mathfrak{s}'}} a_{\mathfrak{u}} U_k m_{\mathfrak{u}\mathfrak{t}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

But $c_{\mathfrak{s}}(k) = c_{\mathfrak{s}'}(k+1)$ and we may conclude this case using the previous Lemma as before. The remaining cases are treated similarly. □

6.   A graded cellular basis of $b_n(m)$.

In this section we obtain our main results showing that $b_n(m)$ is a graded cellular algebra. Our methods are inspired by the ones used by Hu and Mathas in [11, Section 4 and 5], who construct a graded cellular basis $\{\psi_{st}\}$ for the cyclotomic Hecke algebra, in terms of the Khovanov-Lauda-Rouquier generators. But unfortunately is not possible to use their results directly. In fact, the homomorphism $\Phi : \mathcal{H}_n(m) \longrightarrow b_n(m)$ may easily map linearly independent elements to linearly dependent elements. Moreover, due to the incompatibility between the dominance order used for $\{\psi_{st}\}$ and the order $\succ$ for $b_n(m)$, we do not know how to find a basis for $\ker \Phi$ consisting of elements from $\{\psi_{st}\}$, and so in general it seems intractable to determine which are the subsets of $\{\psi_{st}\}$ that stay independent under Φ.

Our solution to this problem is indirect. It is based on an alternative realization of the KLR-idempotents $e(\boldsymbol{i})$ which is possible in the setting of an algebra with JM-elements, see Lemma 4.2 of [17]. It also plays a key role in [11] in the setting of cyclotomic Hecke algebras. To explain it we first setup the relevant notation.

We fix $\mathcal{O}$ and $\mathfrak{m}$ as above. Recall that $K = \mathbb{C}(q, Q)$ and $b_n^K(m) = b_n^{\mathcal{O}}(m) \otimes_{\mathcal{O}} K$. Over K the contents from Definition 5.5 trivially verify the separation criterion of [17] and so $b_n^K(m)$ is semisimple. Hence we can apply [17] to the algebras $b_n(m)$, $b_n^{\mathcal{O}}(m)$ and $b_n^K(m)$. We repeat the necessary definitions in our setting.

Definition 6.1. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we define

$$F_{\mathfrak{t}} := \prod_{k=1}^{n} \prod_{\substack{\mathfrak{s} \in \mathrm{Std}(n) \\ c_{\mathfrak{s}}(k) \neq c_{\mathfrak{s}}(k)}} \frac{L_k - c_{\mathfrak{s}}(k)}{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)} \in b_n^K(m)$$

and set $f_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$.

We extend the order $\succeq$ to pairs of bitableaux of the same shape by declaring $(\mathfrak{u}, \mathfrak{v}) \succeq (\mathfrak{s}, \mathfrak{t})$ if $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu})$, and if either $\boldsymbol{\mu} \succeq \boldsymbol{\lambda}$ or $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\mathfrak{u} \succeq \mathfrak{s}$ and $\mathfrak{v} \succeq \mathfrak{t}$. Then we get that

$$f_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u}, \mathfrak{v}) \succ (\mathfrak{s}, \mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}} \tag{6.1}$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in K$ and hence

$$\{ f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Bip}_1(n) \}$$

is a basis for $b_n^K(m)$, the seminormal basis. Moreover, by [17, Theorem 3.7], for $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ there exists a non-zero scalar $\gamma_{\mathfrak{t}} \in K$ such that

$$f_{\mathfrak{t}\mathfrak{t}} f_{\mathfrak{t}\mathfrak{t}} = \gamma_{\mathfrak{t}} f_{\mathfrak{t}\mathfrak{t}} \tag{6.2}$$

Let $\approx$ be the equivalence relation on $\mathrm{Std}(\boldsymbol{n})$ given by $\mathfrak{s} \approx \mathfrak{t}$ if $r_{\mathfrak{s}}(k) = r_{\mathfrak{s}}(k)$ for $k = 1, 2, \ldots, n$. The equivalence classes for $\approx$ are parametrized by residue sequences I^n of length n; for $\boldsymbol{i} \in I^n$ we denote by $\mathrm{Std}(\boldsymbol{i})$ the corresponding class. Any tableau $\mathfrak{s}$ gives rise to a residue sequence that is denoted $\boldsymbol{i}^{\mathfrak{s}}$. Then we have $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{i}^{\mathfrak{s}})$ but in general $\mathrm{Std}(\boldsymbol{i})$ may be empty, of course. For each $\boldsymbol{i} \in I^n$ we define idempotents $e^b(\boldsymbol{i}) \in b_n^K(m)$ by

$$e^b(\boldsymbol{i}) := \sum_{\mathfrak{s} \in \mathrm{Std}(\boldsymbol{i})} \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}.$$

Then it follows from [17] that actually $e^b(\boldsymbol{i}) \in b_n^{\mathcal{O}}$ and so we may reduce $e^b(\boldsymbol{i})$ modulo $\mathfrak{m}$ to obtain idempotents of $b_n(m)$ that we denote the same way $e^b(\boldsymbol{i})$.

The next result plays a key role in [11] in the setting of cyclotomic Hecke algebras.

Lemma 6.2. *For $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in I^n$ let*

$$b_n(m)(\boldsymbol{i}) := \{v \in b_n(m) \mid (L_r - q^{2i_r})^k v = 0 \text{ for } r = 1, \ldots, n \text{ and } k \gg 0\}$$

be the generalized weight space for the action of $L_i \in b_n(m)$. Then we have $b_n(m)(\boldsymbol{i}) = e^b(\boldsymbol{i})b_n(m)$.

Proof: The proof of Proposition 4.8 of [11] carries over. $\qquad\square$

Lemma 6.3. *Let $\Phi : \mathcal{H}_n(m) \longrightarrow b_n(m)$ be as above and let $\boldsymbol{i} \in I^n$. Then $\Phi(e(\boldsymbol{i})) = e^b(\boldsymbol{i})$. In particular, $e^b(\boldsymbol{i})$ is a homogeneous element of $b_n(m)$ of degree 0.*

Proof: Since Φ is surjective and maps the JM-elements of $\mathcal{H}_n(m)$ to the JM-elements of $b_n(m)$, we have $\Phi(\mathcal{H}_n(m)(\boldsymbol{i})) = b_n(m)(\boldsymbol{i})$. But then

$$e^b(\boldsymbol{i})b_n(m) = b_n(m)(\boldsymbol{i}) = \Phi(\mathcal{H}_n(m)(\boldsymbol{i})) = \Phi(e(\boldsymbol{i})\mathcal{H}_n(m)) = \Phi(e(\boldsymbol{i}))b_n(m).$$

Moreover, $\Phi(e(\boldsymbol{i}))$ lies in the subalgebra of $b_n(m)$ generated by the JM-elements since $e(\boldsymbol{i})$ has the corresponding property, and so $\Phi(e(\boldsymbol{i})) = e^b(\boldsymbol{i})$ as claimed. On the other hand, by Corollary 3.6 we know that Φ is homogeneous and so the second claim holds as well. $\qquad\square$

We next define elements ψ_i^b, y_i^b of $b_n(m)$ by $\psi_i^b := \Phi(\psi_i)$ and $y_i^b := \Phi(y_i)$. As is the case for $e^b(\boldsymbol{i})$, these elements are homogeneous of the same degree as their Hecke algebra counterparts. We are now in position to give the key definition of this section.

Definition 6.4. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Let $\mathfrak{d}(\mathfrak{s}) = s_{i_1} \ldots s_{i_k}$ and $\mathfrak{d}(\mathfrak{t}) = s_{j_1} \ldots s_{j_l}$ be reduced expressions for $\mathfrak{d}(\mathfrak{s})$ and $\mathfrak{d}(\mathfrak{t})$, chosen as in Lemma 4.15. Then we define*

$$\psi_{\mathfrak{s}\mathfrak{t}}^b := \psi_{i_1}^b \ldots \psi_{i_k}^b e^b(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_{j_1}^b \ldots \psi_{j_1}^b \in b_n(m).$$

Note that although our $\psi_{\mathfrak{s}\mathfrak{t}}^b$ look much like the elements $\psi_{\mathfrak{s}\mathfrak{t}}$ introduced in [11], this resemblance is only formal and in general there is no obvious connection between the two families of elements, due to the differences between the tableaux. Note also that in our definition there is no y factor, contrary to the [11] situation. Finally, note that our $\psi_{\mathfrak{s}\mathfrak{t}}^b$ can be shown to be independent of the choices of reduced expressions as above, this is also contrary to the situation in [11].

Our next result is parallel to Theorem 4.14 of [11], but has no y term. This 'missing' y is the reason why there is no y factor in Definition 6.4.

Theorem 6.5. *Suppose that $\boldsymbol{\lambda} = ((a), (b)) \in Bip_1(n)$. Then there exists a nonzero scalar $r \in \mathbb{C}^\times$ such that*

$$e^b(\boldsymbol{i}^{\boldsymbol{\lambda}}) \equiv r m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

Proof: We begin by determining $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$. For this we use (6.1) and (6.2) and find

$$\begin{aligned}
\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} &= f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \\
&\equiv m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} && \mod b_n^{K,\boldsymbol{\lambda}}(m) \\
&\equiv (y_e)^c m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} && \mod b_n^{K,\boldsymbol{\lambda}}(m) \\
&\equiv (y_e)^c f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} && \mod b_n^{K,\boldsymbol{\lambda}}(m)
\end{aligned}$$

where $c = \min\{a, b\}$ and where $m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ can be conveniently found via the diagrammatic realization of $m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ in Figure 6. From this we deduce that $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = (y_e)^c$.

On the other hand, for $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{i^\lambda})$ with $\mathfrak{s} \neq \mathfrak{t}^{\boldsymbol{\lambda}}$, we get by combining the description of $\mathfrak{t}^{\boldsymbol{\lambda}}$ given just after Definition 4.3 with the standing conditions on the parameters (2.4) that $\mathrm{Shape}(\mathfrak{s}) \succ \boldsymbol{\lambda}$. But then (6.1) and the definition of $e(\boldsymbol{i^\lambda})$ imply

$$e(\boldsymbol{i^\lambda}) \equiv \frac{1}{(y_e)^c} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{K,\boldsymbol{\lambda}}(m). \tag{6.3}$$

Since $e(\boldsymbol{i^\lambda})$ and $\frac{1}{(y_e)^c} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ both belong to $b_n^{\mathcal{O},\boldsymbol{\lambda}}(m)$, we can now replace $b_n^{K,\boldsymbol{\lambda}}(m)$ by $b_n^{\mathcal{O},\boldsymbol{\lambda}}(m)$ in (6.3). From this the proof is obtained by reducing modulo $\mathfrak{m}$. $\square$

We can now prove that the elements from Definition 6.4 form a basis for $b_n(m)$.

Theorem 6.6. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then there are scalars $r \in \mathbb{C}^\times$ and $r_{\mathfrak{u}\mathfrak{v}} \in \mathbb{C}$ such that*

$$\psi_{\mathfrak{s}\mathfrak{t}}^b = r m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \succ (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}}.$$

Hence $\{\psi_{\mathfrak{s}\mathfrak{t}}^b \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in Bip_1(n)\}$ is a basis for $b_n(m)$.

Proof: For $\mathfrak{d}(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ a reduced expression for $\mathfrak{d}(\mathfrak{s})$ as above we consider first $\psi_{i_1}^b \dots \psi_{i_k}^b e^b(\boldsymbol{i^\lambda})$. Using (2.9) and the commutation rules (2.17), (2.19) and (2.20) between the y_i and ψ_j, we get that it can be expressed as a linear combination of elements of the form $\Phi(T_{i_{j_1}} \dots T_{i_{j_r}} f_{j_1,\dots,j_r}(y_1, \dots, y_n) e(\boldsymbol{i^\lambda}))$ where $(i_{j_1}, \dots, i_{j_r})$ is a subsequence of $(i_1, \dots, i_k)$ and where $f_{j_1,\dots,j_r}(y_1, \dots, y_n)$ is a polynomial in the y_i. But $\Phi(T_i) = qU_i + q^2$ and hence this can also be written as a linear combination of elements of the form $U_{i_{j_1}} \dots U_{i_{j_r}} g_{j_1,\dots,j_r}(y_1^b, \dots, y_n^b) e^b(\boldsymbol{i^\lambda}))$ where $(i_{j_1}, \dots, i_{j_r})$ is a subsequence of $(i_1, \dots, i_k)$ and $g_{j_1,\dots,j_r}(y_1^b, \dots, y_n^b)$ is a polynomial in the y_i^b. But from (2.8) and the previous Theorem this is a linear combination of elements of the form $U_{i_{j_1}} \dots U_{i_{j_r}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m)$.

Going through the above argument once more, we get that the coefficient of $U_{i_1} \dots U_{i_k} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ in $\psi_{i_1}^b \dots \psi_{i_k}^b e^b(\boldsymbol{i^\lambda})$ is nonzero, in fact it is essentially the product of the constant terms of the polynomials Q appearing in (2.9). But Lemma 5.7 implies, by the choice of reduced expression for $\mathfrak{d}(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$, that $U_{i_1} \dots U_{i_k} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = y_e^l m_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}}$ for some $l \in \mathbb{Z} \geq 0$ and then Lemma 5.11 implies that

$$U_{i_{j_1}} \dots U_{i_{j_r}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = r m_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m)$$

for some scalar $r \in \mathbb{C}$ and some $\mathfrak{u}$ such that $\mathfrak{u} \succ \mathfrak{s}$. Summing up, this proves the Theorem in the case where $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$.

To prove the general case, we first note that the same argument as above, only acting on the right instead of on the left, proves the Theorem in the case where $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$. The general case then follows by multiplying the two versions together and using cellularity. $\square$

Remark 6.7. It follows from the Theorem that the subalgebra of $b_n(m)$ generated by the $e^b(\boldsymbol{i})$ and the ψ_i^b is equal to $b_n(m)$ itself.

To establish our main theorem we must define a degree function on the set of all one-line standard bitableaux. Let $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we define the degree of $\mathfrak{t}$ as

$$\deg \mathfrak{t} := \deg \psi_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}}. \tag{6.4}$$

We can now prove our main result, namely to construct a graded cellular basis for $b_n(m)$. Given our previous work, we can essentially follow the argument of [11, Theorem 5.8], just making the corresponding changes in notation. We sketch the argument because this is the main theorem of the paper.

Theorem 6.8. *The blob algebra $b_n(m)$ is a graded cellular algebra with graded cellular basis $\{\psi^b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in Bip_1(n)\}$.*

Proof: First of all it follows from the triangularity property of Theorem 6.6 that

$$\{\psi^b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$$

is a cellular basis for $b_n(m)$, since $\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$ is it. Moreover, by the definitions, $\psi^b_{\mathfrak{s}\mathfrak{t}}$ is a homogeneous elements of $b_n(m)$ of degree

$$\deg \psi_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}.$$

Using Corollary 3.6 one sees that there is a unique anti-automorphism $*$ of $b_n(m)$ that fixes the generators ψ^b_i, y^b_j and $e^b(\mathbf{i})$. Then by the definition it is clear that $\psi^*_{\mathfrak{s}\mathfrak{t}} = \psi_{\mathfrak{t}\mathfrak{s}}$ and so the anti-automorphism induced by the basis $\{\psi^b_{\mathfrak{s}\mathfrak{t}}\}$ coincides with $*$. The Theorem is proved. $\qquad\square$

By Theorem 6.6 the cell modules induced by the graded cellular bases $\{\psi^b_{\mathfrak{s}\mathfrak{t}}\}$ agree with the cell modules induced by the diagram bases $\{m_{\mathfrak{s}\mathfrak{t}}\}$, that is the standard modules for $b_n(m)$. Therefore, Theorem 6.9 gives us our main goal, to grade the standard modules for $b_n(m)$.

For completeness, we give the analogous Theorem for the Temperley-Lieb algebra. This proof relies here on Theorem 3.2 and the compatibility of Murphy's standard basis with the diagram basis, as proved in [10], and could have been given earlier in the paper. Let $\Phi_2 : \mathcal{H}_n(q^2) \longrightarrow Tl_n(q)$ be as above and define $\psi^{Tl}_{\mathfrak{s}\mathfrak{t}} := \Phi_2(\psi_{st})$ for $s, t \in \mathrm{Std}(n)$ where ψ_{st} is the graded cellular basis for $\mathcal{H}_n(q^2)$ introduced by Hu and Mathas and $\mathrm{Shape}(s) = \mathrm{Shape}(t) \in \mathrm{Par}_2(n)$.

Theorem 6.9. *The Temperley-Lieb algebra $Tl_n(q)$ is a graded cellular algebra with graded cellular basis $\{\psi^{Tl}_{\mathfrak{s}\mathfrak{t}}\}$ and degree function defined as above.*

Proof: According to Theorem 9 of [10], the diagram basis for $Tl_n(q)$ is upper triangularily related to the standard basis, with respect to the dominance order. But ψ_{st} is also upper triangularily related to the standard basis with respect to the dominance order, as already mentioned above, and hence the Theorem follows. $\square$

7. Examples.

In this last section we illustrate our results on two examples.

Example 7.1. Our first example is $Tl_3(q)$, with q chosen to be a primitive cubic root of unity, that is $e = 3$. This is a non-semisimple algebra and so we expect the grading to be nontrivial. Using the homogeneous basis for $\psi^{Tl}_{\mathfrak{s}\mathfrak{t}}$ for $Tl_3(q)$, we aim at determining a homogeneous basis for $Tl_3(q)$, in terms of the diagrams. Define first

$$\sigma = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad\qquad \tau = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Then σ and τ are the only tableaux of shape $(2,1)$. The only other possible shape in $\mathrm{Par}_2(3)$ is $\lambda = (1,1,1)$ whose only standard tableau is t^λ. Hence we get that $Tl_3(q)$ has dimension five with homogeneous basis consisting of the elements

$$\psi_{\mathfrak{ss}}^{Tl}, \psi_{\mathfrak{st}}^{Tl}, \psi_{\mathfrak{ts}}^{Tl}, \psi_{\mathfrak{tt}}^{Tl}, \psi_{t^\lambda t^\lambda}^{Tl}.$$

The residue sequences for σ and τ are $\boldsymbol{i}^\sigma = (0,1,2)$ and $\boldsymbol{i}^\tau = (0,2,1)$ and the degrees are $\deg(\sigma) = 0$ and $\deg(\tau) = 1$. (See [11, (3.8) and Definition 4.7]). Therefore, using the orthogonality of the KLR-idempotents, we have

$$\psi_{\sigma\tau}^{Tl}\psi_{\sigma\sigma}^{Tl} = \psi_{\sigma\sigma}^{Tl}\psi_{\tau\sigma}^{Tl} = 0, \qquad (7.1)$$

see [11, Lemma 5.2]. Now by the triangular expansion property mentioned in the above Theorem 6.9 there exists $c \in \mathbb{C}^\times$ such that

$$\psi_{\sigma\sigma}^{Tl} = c \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} = cU_1$$

and hence U_1 is homogeneous of degree 0. Using the triangular expansion property of Theorem 6.9 once again, there are scalars $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq 0$ such that

$$\psi_{\sigma\tau}^{Tl} = c_1 \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} + c_2 \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}}$$

Multiplying this equality on the right by $\psi_{\sigma\sigma}^{Tl} = cU_1$, and using equation (7.1), we get that $c_1 = [2]c_2$. Hence the element

$$A := [2] \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} + \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}}$$

is a scalar multiple of $\psi_{\sigma\tau}^{Tl}$ and homogeneous of degree 1. Similarly we obtain that the element

$$B := [2] \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} + \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}}$$

is a scalar multiple of $\psi_{\tau\sigma}^{Tl}$ and homogeneous of degree 1.

Now, σ is the maximal tableau of shape $(2,1)$ and so we have $\psi_{\tau\sigma}^{Tl}\psi_{\sigma\tau}^{Tl} = \psi_{\tau\tau}^{Tl}$. From this we obtain that $\psi_{\tau\tau}^{Tl}$ is a scalar multiple of

$$C = \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} + [2] \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} + [2] \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}} + \; \boxed{\begin{array}{c}\smile\\[-4pt]\frown\end{array}}$$

which is a homogeneous element of degree 2.

The last homogeneous basis element can now be determined by expanding $\psi_{t^\lambda t^\lambda}^{Tl}$ in the diagram basis. On the other hand, since the unity 1 is always homogeneous of degree 0 and since it is linearly independent of A, B, C and U_1, we use it. All in all, the set $\{1, U_1, A, B, C\}$ is a homogeneous bases for $Tl_3(q)$. In particular, $Tl_3(q)$ is a positively graded algebra and $\mathcal{F}_1 := \mathrm{span}_\mathbb{C}\{A, B, C\}$ and $\mathcal{F}_2 := \mathrm{span}_\mathbb{C}\{C\}$ are ideals in $Tl_3(q)$. In general, $Tl_n(q)$ is not positively graded.

Example 7.2. We now describe a homogeneous basis for $b_3 = b_3(q, y_e)$ in terms of blob diagrams, with q a primitive quintic root of unity and $y_e = -\frac{[1]}{[2]}$, so in this case $e = 5$ and $m = 2$. First, we list all elements in $\mathrm{Std}(3)$, with their respective residues sequences and degrees.

Bi-partitions	Bitableaux	Res. Sequence	Degree
$\boldsymbol{\lambda} = ((1),(2))$	$\mathfrak{t}^{\boldsymbol{\lambda}} = (\,\boxed{2}\,,\boxed{1}\boxed{3}\,)$	$\boldsymbol{i}^{\boldsymbol{\lambda}} = (4,1,0)$	0
	$\mathfrak{s} = (\,\boxed{3}\,,\boxed{1}\boxed{2}\,)$	$\boldsymbol{i}^{\mathfrak{s}} = (4,0,1)$	1
	$\mathfrak{t} = (\,\boxed{1}\,,\boxed{2}\boxed{3}\,)$	$\boldsymbol{i}^{\mathfrak{t}} = (1,4,0)$	0
$\boldsymbol{\mu} = ((2),(1))$	$\mathfrak{t}^{\boldsymbol{\mu}} = (\,\boxed{2}\boxed{3}\,,\boxed{1}\,)$	$\boldsymbol{i}^{\boldsymbol{\mu}} = (4,1,2)$	0
	$\mathfrak{v} = (\,\boxed{1}\boxed{3}\,,\boxed{2}\,)$	$\boldsymbol{i}^{\mathfrak{v}} = (1,4,2)$	0
	$\mathfrak{u} = (\,\boxed{1}\boxed{2}\,,\boxed{3}\,)$	$\boldsymbol{i}^{\mathfrak{u}} = (1,2,4)$	0
$\boldsymbol{\nu} = ((0),(3))$	$\mathfrak{t}^{\boldsymbol{\nu}} = (\emptyset, \boxed{1}\boxed{2}\boxed{3}\,)$	$\boldsymbol{i}^{\boldsymbol{\nu}} = (4,0,1)$	0
$\boldsymbol{\kappa} = ((3),(0))$	$\mathfrak{t}^{\boldsymbol{\kappa}} = (\,\boxed{1}\boxed{2}\boxed{3}\,,\emptyset)$	$\boldsymbol{i}^{\boldsymbol{\kappa}} = (1,2,3)$	0

We need the following Lemma.

Lemma 7.3. *Let $k \in \mathbb{Z}$ such that $2k \equiv m \mod e$. Then the element $e \in b_n(m)$ is homogeneous of degree zero. More precisely, it can be written as a sum of homogeneous elements of degree zero as follows.*

$$e = \sum_{\substack{\boldsymbol{i} \in I^n \\ i_1 = -k}} e(\boldsymbol{i}) \tag{7.2}$$

Furthermore, for all $\mathfrak{s}, \mathfrak{t} \in Std(n)$ we have

$$e\psi_{\mathfrak{s}\mathfrak{t}}^b e = \begin{cases} \psi_{\mathfrak{s}\mathfrak{t}}^b, & \text{if } 1 \text{ is located in the second component of } \mathfrak{s} \text{ and } \mathfrak{t}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: The first claim follows by combining Theorem 2.6, (2.23) and $U_0 = -[m]e$. The second is a direct consequence of the definition of $\psi_{\mathfrak{s}\mathfrak{t}}^b$, the orthogonality of KLR-idempotents and the first claim. $\qquad\square$

Using the triangularity property given in Lemma 6.6, the orthogonality of the KLR-idempotents and the previous Lemma 7.3, we can now obtain a homogeneous basis for b_3 in terms of diagrams arguing as in the previous example. We omit the details for brevity, since $\dim_{\mathbb{C}}(b_3)$ is quite big, of dimension 20.

$$A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = \quad A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} = \quad A_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} = \quad A_{\mathfrak{t}\mathfrak{t}} =$$

$$A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} = y_e \quad - \quad \qquad A_{\mathfrak{t}\mathfrak{s}} = y_e \quad -$$

$$A_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} = y_e \quad - \quad \qquad A_{\mathfrak{s}\mathfrak{t}} = y_e \quad -$$

$$A_{\mathfrak{s}\mathfrak{s}} = y_e^2 \quad - y_e \quad - y_e \quad +$$

$$A_{\mathfrak{t}^\mu \mathfrak{t}^\mu} = \quad A_{\mathfrak{t}^\mu \mathfrak{v}} = \quad A_{\mathfrak{v} \mathfrak{t}^\mu} = \quad A_{\mathfrak{v} \mathfrak{v}} =$$

$$A_{\mathfrak{t}^\mu \mathfrak{u}} = \quad - \quad \quad A_{\mathfrak{v} \mathfrak{u}} = \quad -$$

$$A_{\mathfrak{u} \mathfrak{t}^\mu} = \quad - \quad \quad A_{\mathfrak{u} \mathfrak{v}} = \quad -$$

$$A_{\mathfrak{u} \mathfrak{u}} = \quad - \quad \quad A_{\mathfrak{t}^\nu \mathfrak{t}^\nu} = \quad \quad A_{\mathfrak{t}^\kappa \mathfrak{t}^\kappa} =$$

The set $\{A_{\mathfrak{a}\mathfrak{b}} \mid \mathfrak{a}, \mathfrak{b} \in \mathrm{Std}(\boldsymbol{\lambda}) \ , \ \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$ is a homogeneous basis for b_3 with $\deg(A_{\mathfrak{a}\mathfrak{b}}) = \deg(\mathfrak{a}) + \deg(\mathfrak{b})$. We remark that the elements $A_{\mathfrak{a}\mathfrak{b}}$ in general do not coincide with the elements $\psi^b_{\mathfrak{a}\mathfrak{b}}$. We also remark that, just like in the Temperley-Lieb algebra case, in general the blob algebra is not positively graded.

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